HYDRODYNAMICS

CHAPTER I

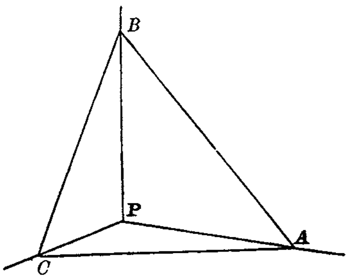
THE EQUATIONS OF MOTION

1. THE following investigations proceed on the assumption that the matter with which we deal may be treated as practically continuous and homogeneous in structure; *i.e.* we assume that the properties of the smallest portions into which we can conceive it to be divided are the same as those of the substance in bulk.

The fundamental property of a fluid is that it cannot be in equilibrium in a state of stress such that the mutual action between two adjacent parts is oblique to the common surface. This property is the basis of Hydrostatics, and is verified by the complete agreement of the deductions of that science with experiment. Very slight observation is enough, however, to convince us that oblique stresses may exist in fluids *in motion*. Let us suppose for instance that a vessel in the form of a circular cylinder, containing water (or other liquid), is made to rotate about its axis, which is vertical. If the angular velocity of the vessel be constant, the fluid is soon found to be rotating with the vessel as one solid body. If the vessel be now brought to rest, the motion of the fluid continues for some time, but gradually subsides, and at length ceases altogether; and it is found that during this process the portions of fluid which are further from the axis lag behind those which are nearer, and have their motion more rapidly checked. These phenomena point to the existence of mutual actions between contiguous elements which are partly tangential to the common surface. For if the mutual action were everywhere wholly normal, it is obvious that the moment of momentum, about the axis of the vessel, of any portion of fluid bounded by a surface of revolution about this axis, would be constant. We infer, moreover, that these tangential stresses are not called into play so long as the fluid moves as a solid body, but only whilst a change of shape of some portion of the mass is going on, and that their tendency is to oppose this change of shape.

2. It is usual, however, in the first instance to neglect the tangential stresses altogether. Their effect is in many practical cases small, and, independently of this, it is convenient to divide the not inconsiderable difficulties of our subject by investigating first the effects of purely normal stress. The further consideration of the laws of tangential stress is accordingly deferred till Chapter XI.

If the stress exerted across any small plane area situate at a point P of the fluid be wholly normal, its intensity (per unit area) is the same for all aspects of the plane. The following proof of this theorem is given here for purposes of reference. Through P draw three straight lines PA, PB, PC mutually at right angles, and let a plane whose direction-cosines relatively to these lines are l, m, n, passing infinitely close to P, meet them in A, B, C. Let p, p_1, p_2, p_3 denote the intensities of the stresses* across the faces ABC, PBC, PCA, PAB, respectively, of the tetrahedron $PABC$. If Δ be the area of the first-mentioned face, the areas of the others are, in order, $l\Delta$, $m\Delta$, $n\Delta$. Hence if we form the equation of motion of the tetrahedron parallel to PA we have $p_1 \,.\, l\Delta = pl \,.\, \Delta$, where we have omitted the terms which express the rate of change of momentum, and the component of the extraneous forces, because they are ultimately proportional to the mass of the tetrahedron, and therefore of the third order of small linear quantities, whilst the terms retained are of the second. We have then, ultimately, $p = p_1$, and similarly $p = p_2 = p_3$, which proves the theorem.



3. The equations of motion of a fluid have been obtained in two different forms, corresponding to the two ways in which the problem of determining the motion of a fluid mass, acted on by given forces and subject to given conditions, may be viewed. We may either regard as the object of our investigations a knowledge of the velocity, the pressure, and the density, at all points of space occupied by the fluid, for all instants; or we may seek to determine the history of every particle. The equations obtained on these two plans are conveniently designated, as by German mathematicians, the 'Eulerian' and the 'Lagrangian' forms of the hydrokinetic equations, although both forms are in reality due to Euler†.

The Eulerian Equations.

4. Let u, v, w be the components, parallel to the co-ordinate axes, of the velocity at the point (x, y, z) at the time t. These quantities are then functions of the independent variables x, y, z, t. For any particular value of t they define the motion at that instant at all points of space occupied by

* Reckoned positive when pressures, negative when tensions. Most fluids are, however, incapable under ordinary conditions of supporting more than an exceedingly slight degree of tension, so that p is nearly always positive.

† "Principes généraux du mouvement des fluides," *Hist. de l'Acad. de Berlin*, 1755.
"De principiis motus fluidorum," *Novi Comm. Acad. Petrop.* xiv. 1 (1759).
Lagrange gave three investigations of the equations of motion; first, incidentally, in

the fluid; whilst for particular values of x, y, z they give the history of what goes on at a particular place.

We shall suppose, for the most part, not only that u, v, w are finite and continuous functions of x, y, z, but that their space-derivatives of the first order ($\partial u/\partial x$, $\partial v/\partial x$, $\partial w/\partial x$, &c.) are everywhere finite*; we shall understand by the term 'continuous motion,' a motion subject to these restrictions. Cases of exception, if they present themselves, will require separate examination. In continuous motion, as thus defined, the relative velocity of any two neighbouring particles P, P' will always be infinitely small, so that the line PP' will always remain of the same order of magnitude. It follows that if we imagine a small closed surface to be drawn, surrounding P, and suppose it to move with the fluid, it will always enclose the same matter. And *any* surface whatever, which moves with the fluid, completely and permanently separates the matter on the two sides of it.

5. The values of u, v, w for successive values of t give as it were a series of pictures of consecutive stages of the motion, in which however there is no immediate means of tracing the identity of any one particle.

To calculate the rate at which any function $F(x, y, z, t)$ varies for a moving particle, we may remark that at the time $t + \delta t$ the particle which was originally in the position (x, y, z) is in the position $(x + u\,\delta t,\ y + v\,\delta t,\ z + w\,\delta t)$, so that the corresponding value of F is

$$F(x + u\,\delta t,\ y + v\,\delta t,\ z + w\,\delta t,\ t + \delta t) = F + u\,\delta t\,\frac{\partial F}{\partial x} + v\,\delta t\,\frac{\partial F}{\partial y} + w\,\delta t\,\frac{\partial F}{\partial z} + \delta t\,\frac{\partial F}{\partial t}.$$

If, after Stokes, we introduce the symbol D/Dt to denote a differentiation following the motion of the fluid, the new value of F is also expressed by $F + DF/Dt\,.\,\delta t$, whence

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z}. \quad \text{......................(1)}$$

6. To form the dynamical equations, let p be the pressure, ρ the density, X, Y, Z the components of the extraneous forces per unit mass, at the point (x, y, z) at the time t. Let us take an element having its centre at (x, y, z), and its edges $\delta x, \delta y, \delta z$ parallel to the rectangular co-ordinate axes. The rate at which the x-component of the momentum of this element is increasing is $\rho\,\delta x\,\delta y\,\delta z\,Du/Dt$; and this must be equal to the x-component of the forces

connection with the principle of Least Action, in the *Miscellanea Taurinensia*, ii. (1760) [*Oeuvres*, Paris, 1867–92, i.]; secondly in his "Mémoire sur la Théorie du Mouvement des Fluides," *Nouv. mém. de l'Acad. de Berlin*, 1781 [*Oeuvres*, iv.]; and thirdly in the *Mécanique Analytique*. In this last exposition he starts with the second form of the equations (Art. 14, below), but translates them at once into the 'Eulerian' notation.

* It is important to bear in mind, with a view to some later developments under the head of Vortex Motion, that these derivatives need not be assumed to be continuous.

acting on the element. Of these the extraneous forces give $\rho\,\delta x\,\delta y\,\delta z\,X$. The pressure on the yz-face which is nearest the origin will be ultimately

$$(p - \tfrac{1}{2}\partial p/\partial x\,.\,\delta x)\,\delta y\,\delta z^*,$$

that on the opposite face

$$(p + \tfrac{1}{2}\partial p/\partial x\,.\,\delta x)\,\delta y\,\delta z.$$

The difference of these gives a resultant $-\partial p/\partial x\,.\,\delta x\,\delta y\,\delta z$ in the direction of x-positive. The pressures on the remaining faces are perpendicular to x. We have then

$$\rho\,\delta x\,\delta y\,\delta z\,\frac{Du}{Dt} = \rho\,\delta x\,\delta y\,\delta z\,X - \frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z.$$

Substituting the value of Du/Dt from (1), and writing down the symmetrical equations, we have

$$\left.\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= X - \frac{1}{\rho}\frac{\partial p}{\partial x},\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= Y - \frac{1}{\rho}\frac{\partial p}{\partial y},\\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= Z - \frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\} \quad\ldots\ldots\ldots\ldots\ldots(2)$$

7. To these dynamical equations we must join, in the first place, a certain kinematical relation between u, v, w, ρ, obtained as follows.

If Q be the volume of a moving element, we have, on account of the constancy of mass,

$$\frac{D\,.\,\rho Q}{Dt} = 0,$$

or

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{1}{Q}\frac{DQ}{Dt} = 0. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

To calculate the value of $1/Q\,.\,DQ/Dt$, let the element in question be that which at time t fills the rectangular space $\delta x\,\delta y\,\delta z$ having one corner P at (x, y, z), and the edges PL, PM, PN (say) parallel to the co-ordinate axes. At time $t + \delta t$ the same element will form an oblique parallelepiped, and since the velocities of the particle L relative to the particle P are $\partial u/\partial x\,.\,\delta x$, $\partial v/\partial x\,.\,\delta x$, $\partial w/\partial x\,.\,\delta x$, the projections of the edge PL on the co-ordinate axes become, after the time δt,

$$\left(1 + \frac{\partial u}{\partial x}\delta t\right)\delta x, \qquad \frac{\partial v}{\partial x}\delta t\,.\,\delta x, \qquad \frac{\partial w}{\partial x}\delta t\,.\,\delta x,$$

respectively. To the first order in δt, the length of this edge is now

$$\left(1 + \frac{\partial u}{\partial x}\delta t\right)\delta x,$$

and similarly for the remaining edges. Since the angles of the parallelepiped

* It is easily seen, by Taylor's theorem, that the mean pressure over any face of the element $\delta x\,\delta y\,\delta z$ may be taken to be equal to the pressure at the centre of that face.

differ infinitely little from right angles, the volume is still given, to the first order in δt, by the product of the three edges, *i.e.* we have

$$Q + \frac{DQ}{Dt}\delta t = \left\{1 + \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\delta t\right\}\delta x\,\delta y\,\delta z,$$

or

$$\frac{1}{Q}\frac{DQ}{Dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}. \qquad \text{.........................(2)}$$

Hence (1) becomes

$$\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0. \qquad \text{.....................(3)}$$

This is called the 'equation of continuity.'

The expression

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}, \qquad \text{..............................(4)}$$

which, as we have seen, measures the rate of dilatation of the fluid at the point (x, y, z), is conveniently called the 'expansion' at that point. From a more general point of view the expression (4) is called the 'divergence' of the vector (u, v, w); it is often denoted briefly by

$$\text{div}\,(u, v, w).$$

The preceding investigation is substantially that given by Euler*. Another, and now more usual, method of obtaining the equation of continuity is, instead of following the motion of a fluid element, to fix the attention on an element $\delta x\,\delta y\,\delta z$ of space, and to calculate the change produced in the included mass by the flux across the boundary. If the centre of the element be at (x, y, z), the amount of matter which per unit time enters it across the yz-face nearest the origin is

$$\left(\rho u - \tfrac{1}{2}\frac{\partial \cdot \rho u}{\partial x}\delta x\right)\delta y\,\delta z,$$

and the amount which leaves it by the opposite face is

$$\left(\rho u + \tfrac{1}{2}\frac{\partial \cdot \rho u}{\partial x}\delta x\right)\delta y\,\delta z.$$

The two faces together give a gain

$$-\frac{\partial \cdot \rho u}{\partial x}\delta x\,\delta y\,\delta z,$$

per unit time. Calculating in the same way the effect of the flux across the remaining faces, we have for the total gain of mass, per unit time, in the space $\delta x\,\delta y\,\delta z$, the formula

$$-\left(\frac{\partial \cdot \rho u}{\partial x} + \frac{\partial \cdot \rho v}{\partial y} + \frac{\partial \cdot \rho w}{\partial z}\right)\delta x\,\delta y\,\delta z.$$

Since the quantity of matter in any region can vary only in consequence of the flux across the boundary, this must be equal to

$$\frac{\partial}{\partial t}(\rho\,\delta x\,\delta y\,\delta z),$$

* *l.c. ante* p. 2.

whence we get the equation of continuity in the form

$$\frac{\partial\rho}{\partial t}+\frac{\partial\,.\,\rho u}{\partial x}+\frac{\partial\,.\,\rho v}{\partial y}+\frac{\partial\,.\,\rho w}{\partial z}=0. \quad \text{.....................(5)}$$

8. It remains to put in evidence the physical properties of the fluid, so far as these affect the quantities which occur in our equations.

In an 'incompressible' fluid, or liquid, we have $D\rho/Dt=0$, in which case the equation of continuity takes the simple form

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \quad \text{..............................(1)}$$

It is not assumed here that the fluid is of *uniform* density, though this is of course by far the most important case.

If we wish to take account of the slight compressibility of actual liquids, we shall have a relation of the form

$$p=\kappa(\rho-\rho_0)/\rho_0, \quad \text{............................(2)}$$

or

$$\rho/\rho_0=1+p/\kappa, \quad \text{............................(3)}$$

where κ denotes what is called the 'elasticity of volume.'

In the case of a gas whose temperature is uniform and constant we have the 'isothermal' relation

$$p/p_0=\rho/\rho_0, \quad \text{................................(4)}$$

where p_0, ρ_0 are any pair of corresponding values for the temperature in question.

In most cases of motion of gases, however, the temperature is not constant, but rises and falls, for each element, as the gas is compressed or rarefied. When the changes are so rapid that we can ignore the gain or loss of heat by an element due to conduction and radiation, we have the 'adiabatic' relation

$$p/p_0=(\rho/\rho_0)^\gamma, \quad \text{............................(5)}$$

where p_0 and ρ_0 are any pair of corresponding values for the element considered. The constant γ is the ratio of the two specific heats of the gas; for atmospheric air, and some other gases, its value is about 1·408.

9. At the boundaries (if any) of the fluid, the equation of continuity is replaced by a special surface-condition. Thus at a *fixed* boundary, the velocity of the fluid perpendicular to the surface must be zero, *i.e.* if l, m, n be the direction-cosines of the normal,

$$lu+mv+nw=0. \quad \text{............................(1)}$$

Again at a surface of discontinuity, *i.e.* a surface at which the values of u, v, w change abruptly as we pass from one side to the other, we must have

$$l(u_1-u_2)+m(v_1-v_2)+n(w_1-w_2)=0, \quad \text{...............(2)}$$

where the suffixes are used to distinguish the values on the two sides. The same relation must hold at the common surface of a fluid and a moving solid.

The general surface-condition, of which these are particular cases, is that if $F(x, y, z, t)=0$ be the equation of a bounding surface, we must have at every point of it

$$DF/Dt=0. \qquad \text{.................................(3)}$$

For the velocity relative to the surface of a particle lying in it must be wholly tangential (or zero), otherwise we should have a finite flow of fluid across it. It follows that the instantaneous rate of variation of F for a surface-particle must be zero.

A fuller proof, given by Lord Kelvin*, is as follows. To find the rate of motion (ν) of the surface $F(x, y, z, t)=0$, normal to itself, we write

$$F(x+l\nu\,\delta t,\ \ y+m\nu\,\delta t,\ \ z+n\nu\,\delta t,\ \ t+\delta t)=0,$$

where l, m, n are the direction-cosines of the normal at (x, y, z). Hence

$$\nu\left(l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}\right)+\frac{\partial F}{\partial t}=0$$

Since
$$(l, m, n)=\left(\frac{\partial F}{\partial x},\ \frac{\partial F}{\partial y},\ \frac{\partial F}{\partial z}\right)\div R,$$

where
$$R=\left\{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2\right\}^{\frac{1}{2}},$$

we have
$$\nu=-\frac{1}{R}\frac{\partial F}{\partial t}. \qquad \text{...(4)}$$

At every point of the surface we must have

$$\nu=lu+mv+nw,$$

which leads, on substitution of the above values of l, m, n, to the equation (3).

The partial differential equation (3) is also satisfied by any surface moving with the fluid. This follows at once from the meaning of the operator D/Dt. A question arises as to whether the converse necessarily holds; *i.e.* whether a moving surface whose equation $F=0$ satisfies (3) will always consist of the same particles. Considering any such surface, let us fix our attention on a particle P situate on it at time t. The equation (3) expresses that the rate at which P is separating from the surface is at this instant zero: and it is easily seen that *if the motion be continuous* (according to the definition of Art. 4), the normal velocity, relative to the moving surface F, of a particle at an infinitesimal distance ζ from it is of the order ζ, viz. it is equal to $G\zeta$ where G is finite. Hence the equation of motion of the particle P relative to the surface may be written

$$D\zeta/Dt=G\zeta.$$

This shews that $\log\zeta$ increases at a finite rate, and since it is negative infinite to begin with (when $\zeta=0$), it remains so throughout, *i.e.* ζ remains zero for the particle P.

The same result follows from the nature of the solution of

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0, \qquad \text{.................................(5)}$$

considered as a partial differential equation in F†. The subsidiary system of ordinary differential equations is

$$dt=\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}, \qquad \text{......................................(6)}$$

* (W. Thomson) "Notes on Hydrodynamics," *Camb. and Dub. Math. Journ.* Feb. 1848. [*Mathematical and Physical Papers*, Cambridge, 1882..., i. 83.]

† Lagrange, *Oeuvres*, iv. 706.

in which x, y, z are regarded as functions of the independent variable t. These are evidently the equations to find the paths of the particles, and their integrals may be supposed put in the forms

$$x = f_1(a, b, c, t),\quad y = f_2(a, b, c, t),\quad z = f_3(a, b, c, t), \quad \text{.................(7)}$$

where the arbitrary constants a, b, c are any three quantities serving to identify a particle; for instance they may be the initial co-ordinates. The general solution of (5) is then found by elimination of a, b, c between (7) and

$$F = \psi(a, b, c), \quad \text{..(8)}$$

where ψ is an arbitrary function. This shews that a particle once in the surface $F = 0$ remains in it throughout the motion.

Equation of Energy.

10. In most cases which we shall have occasion to consider the extraneous forces have a potential; viz. we have

$$X, Y, Z = -\frac{\partial\Omega}{\partial x},\ -\frac{\partial\Omega}{\partial y},\ -\frac{\partial\Omega}{\partial z}. \quad \text{....................(1)}$$

The physical meaning of Ω is that it denotes the potential energy, per unit mass, at the point (x, y, z), in respect of forces acting at a distance. It will be sufficient for the present to consider the case where the field of extraneous force is constant with respect to the time, *i.e.* $\partial\Omega/\partial t = 0$. If we now multiply the equations (2) of Art. 6 by u, v, w, in order, and add, we obtain a result which may be written

$$\tfrac{1}{2}\rho\frac{D}{Dt}(u^2+v^2+w^2) + \rho\frac{D\Omega}{Dt} = -\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right).$$

If we multiply this by $\delta x\,\delta y\,\delta z$, and integrate over any region, we find

$$\frac{D}{Dt}(T+V) = -\iiint\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z}\right)dx\,dy\,dz, \quad \text{.........(2)}$$

where $\quad T = \tfrac{1}{2}\iiint\rho(u^2+v^2+w^2)\,dx\,dy\,dz, \qquad V = \iiint\Omega\rho\,dx\,dy\,dz, \quad$ ......(3)

i.e. T and V denote the kinetic energy and the potential energy in relation to the field of extraneous force, of the fluid which at the moment occupies the region in question. The triple integral on the right-hand side of (2) may be transformed by a process which will often recur in our subject. Thus, by a partial integration,

$$\iiint u\frac{\partial p}{\partial x}\,dx\,dy\,dz = \iint[pu]\,dy\,dz - \iiint p\frac{\partial u}{\partial x}\,dx\,dy\,dz,$$

where $[pu]$ is used to indicate that the values of pu at the points where the boundary of the region is met by a line parallel to x are to be taken, with proper signs. If l, m, n be the direction-cosines of the *inwardly* directed normal to any element δS of this boundary, we have $\delta y\,\delta z = \pm\, l\,\delta S$, the signs alternating at the successive intersections referred to. We thus find that

$$\iint[pu]\,dy\,dz = -\iint pul\,dS,$$

where the integration extends over the whole bounding surface. Transforming the remaining terms in a similar manner, we obtain

$$\frac{D}{Dt}(T+V)=\iint p\,(lu+mv+nw)\,dS+\iiint p\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)dx\,dy\,dz. \quad \text{...(4)}$$

In the case of an incompressible fluid this reduces to the form

$$\frac{D}{Dt}(T+V)=\iint (lu+mv+nw)\,p\,dS. \quad \text{..................(5)}$$

Since $lu+mv+nw$ denotes the velocity of a fluid particle in the direction of the normal, the latter integral expresses the rate at which the pressures $p\,\delta S$ exerted from without on the various elements δS of the boundary are doing work. Hence the total increase of energy, kinetic and potential, of any portion of the liquid, is equal to the work done by the pressures on its surface.

In particular, if the fluid be bounded on all sides by fixed walls, we have

$$lu+mv+nw=0$$

over the boundary, and therefore

$$T+V=\text{const.} \quad \text{...........................(6)}$$

A similar interpretation can be given to the more general equation (4), provided p be a function of ρ only. If we write

$$E=-\int p\,d\left(\frac{1}{\rho}\right), \quad \text{...........................(7)}$$

then E measures the work done by unit mass of the fluid against external pressure, as it passes, under the supposed relation between p and ρ, from its actual volume to some standard volume. For example, if the unit mass were enclosed in a cylinder with a sliding piston of area A, then when the piston is pushed outwards through a space δx, the work done is $pA\,.\,\delta x$, of which the factor $A\,\delta x$ denotes the increment of volume, *i.e.* of ρ^{-1}. In the case of the adiabatic relation we find

$$E=\frac{1}{\gamma-1}\left(\frac{p}{\rho}-\frac{p_0}{\rho_0}\right). \quad \text{.........................(8)}$$

We may call E the intrinsic energy of the fluid, per unit mass. Now, recalling the interpretation of the expression

$$\partial u/\partial x+\partial v/\partial y+\partial w/\partial z,$$

given in Art. 7, we see that the volume-integral in (4) measures the rate at which the various elements of the fluid are losing intrinsic energy by expansion; it is therefore equal to $-DW/Dt$,

where

$$W=\iiint E\rho\,dx\,dy\,dz. \quad \text{...........................(9)}$$

Hence

$$\frac{D}{Dt}(T+V+W)=\iint p\,(lu+mv+nw)\,dS. \quad \text{...........(10)}$$

The total energy, which is now partly kinetic, partly potential in relation to a constant field of force, and partly intrinsic, is therefore increasing at a rate equal to that at which work is being done on the boundary by pressure from without.

On the isothermal hypothesis we should have

$$E = c^2 \log(\rho/\rho_0), \qquad\qquad (11)$$

where $c^2 = p_0/\rho_0$. This measures the 'free energy' per unit mass. With this definition of E we have an equation of the same *form* as (10), although the meaning is different.

Transfer of Momentum.

10 a. If we fix our attention on the fluid which at the instant t occupies a certain region, the space which it occupies after a time δt will differ from the original region by the addition of a surface film of (positive or negative) thickness

$$(lu + mv + nw)\,\delta t,$$

where (l, m, n) is the direction of the outward normal to the surface. Hence it is easy to see that the rate, at time t, at which the momentum of this particular portion of fluid is increasing is equal to the rate of increase of the momentum contained in a *fixed* region having the same boundary, together with the flux of momentum outwards across the boundary.

In symbols, considering momentum parallel to Ox, we have

$$\begin{aligned}\iiint \frac{Du}{Dt}\rho\, dx\,dy\,dz &= \iiint \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) dx\,dy\,dz \\ &= \iiint \rho\frac{\partial u}{\partial t}\, dx\,dy\,dz + \iint \rho u\,(lu + mv + nw)\, dS \\ &\qquad - \iiint u\left(\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right) dx\,dy\,dz \\ &= \frac{d}{dt}\iiint \rho u\, dx\,dy\,dz + \iint \rho u\,(lu + mv + nw)\, dS, \qquad (1)\end{aligned}$$

by Art. 7 (5).

In steady motion (Art. 21) the first term on the right hand disappears, and the rate of increase of momentum of any portion of fluid is equal to the flux of momentum outwards across its boundary.

Conversely, if we apply the above principle to the fluid contained at any instant in a rectangular space $\delta x\,\delta y\,\delta z$, we reproduce the equation of motion (Art. 6).

Impulsive Generation of Motion.

11. If at any instant impulsive forces act bodily on the fluid, or if the boundary conditions suddenly change, a sudden alteration in the motion may take place. The latter case may arise, for instance, when a solid immersed in the fluid is suddenly set in motion.