CHAPTER I

Preliminary Results

Here we give general results about finite groups, mainly solvable groups and p-groups, including some special properties of groups of odd order. In Chapters II–IV we will apply the results of this section to a hypothetical minimal counterexample to the Odd Order Theorem. As mentioned in the preface, all necessary references in this chapter are taken from **G**.

1. Elementary Properties of Solvable Groups

Suppose G is a group. We say that a group A *operates* on G, or A is an *operator group* on G, if there is given a homomorphism ϕ from A into $\operatorname{Aut} G$. In this case we usually write x^α instead of $\phi(\alpha)(x)$ for $x \in G$ and $\alpha \in A$. We say that A *fixes* an element x of G, or that x is A-*invariant*, if $x^\alpha = x$ for every $\alpha \in A$. We say that A *fixes* a nonempty subset S of G, or that S is A-*invariant*, if each element of A fixes S as a set. As in **G**, pp. 30, 33, the set (group) of all A-invariant elements of G will be denoted by $C_G(A)$. Similarly, if S is a nonempty subset of G, $C_A(S)$ will denote the set of all elements of A that fix every element of S.

We will frequently use the fact (**G**, p. 18) that if H and K are subgroups of a group G, then

$$[H, K] \lhd \langle H, K \rangle.$$

By applying this fact to the semidirect product of a group G by an operator group A, we see that $[G, A]$ is a normal subgroup of G fixed by A. As in **G**, p. 19, $[G, A, A]$ will denote $[[G, A], A]$. Also, we say A *stabilizes* a normal series

$$G = G_0 \supseteq G_1 \supseteq \cdots \supseteq G_n = 1$$

of G if each G_i is A-invariant and A acts trivially on each factor G_{i-1}/G_i, $1 \le i \le n$.

Suppose that A is an operator group on a group G. As in **FT**, p. 840, we say that A acts in a *prime manner* on G if

$$C_G(\alpha) = C_G(A) \text{ for all } \alpha \in A^{\#}.$$

(Note that this must occur if $|A|$ is prime and that we allow $A = 1$.) We say that A acts *regularly*, or in a *regular manner* on G if

$$C_G(\alpha) = 1 \text{ for all } \alpha \in A^{\#}.$$

(Thus, if A acts regularly, then $A \subseteq \operatorname{Aut} G$ and A acts in a prime manner on G. This disagrees slightly with the definition in **G**, p. 39, which requires also that $A \neq 1$.)

In the subsequent text we will write $H \lhd\lhd G$ to mean that H is a *subnormal* subgroup of G. This means that H is a member of a normal series of G (**G**, Exercise 1.5, p. 13). Equivalently, there exists a series

$$H = H_0 \lhd H_1 \lhd \cdots \lhd H_n = G.$$

We use the property that every subgroup of a nilpotent group G is subnormal in G. This follows immediately from the fact that proper subgroups of a nilpotent group are properly contained in their normalizers (**G**, Theorem 2.3.4, p. 22).

All groups considered in this work will be finite except when explicitly stated otherwise.

For later use we make the following definition. Given a prime p and a group G, we say that G has *p-length one* if $G = O_{p',p,p'}(G)$. (This differs slightly from the definition in **G**, p. 227, in that our definition includes groups of p'-order, that is, groups that, according to the usual definition, would have p-length zero.)

A group G is called a *Z-group* if all of its Sylow subgroups are cyclic.

For any subset T of G we define

$$\mathscr{C}_G(T) = \{\, t^g \mid t \in T \text{ and } g \in G \,\}.$$

A nonempty subset X of G is a *TI-subset* of G if $X \cap X^g \subseteq 1$ for all $x \in G - N(X)$. In particular, a nonidentity proper subgroup H of G is a *TI-subgroup* of G if $H \cap H^g = 1$ for all $g \in G - N(H)$.

In the text that follows we will denote by $\mathcal{E}_p(G)$ the set of all elementary abelian p-subgroups of G; $\mathcal{E}_p{}^*(G)$ the set of all maximal elementary abelian p-subgroups of G; and $\mathcal{E}_p{}^i(G)$ the set of all elementary abelian subgroups of order p^i in G (where i is a positive integer). We let $\mathcal{E}(G)$ be the union of the sets $\mathcal{E}_q(G)$ for all primes q. We define $\mathcal{E}^*(G)$ and $\mathcal{E}^i(G)$ analogously.

For a prime p, a p-group R will be called *narrow* if it contains no elementary abelian subgroup of order p^3 or if it contains a subgroup R_0 of order p and a cyclic subgroup R_1 such that $C_R(R_0) = R_0 \times R_1$. (This definition is not standard and is used only in this book. It corresponds to the definition of π^* on p. 845 of **FT**.)

Lemma 1.1. Suppose that M is a minimal normal subgroup of a finite group G. If M is solvable, then $M \subseteq Z(F(G))$ and is elementary abelian.

Proof. Elementary. □

Proposition 1.2 (P. Hall). Suppose that G is a solvable group and that $G^* \lhd G$. Let $\mathscr{D}$ be the set of all chief factors U/V of G. Let $\mathscr{D}^*$ be the set of all chief factors U/V of G for which $U \subseteq F(G^*)$. Then

$$F(G^*) = \bigcap_{U/V \in \mathscr{D}} C_{G^*}(U/V) = \bigcap_{U/V \in \mathscr{D}^*} C_{G^*}(U/V).$$

Proof. Let

$$H = \bigcap_{U/V \in \mathscr{D}} C_{G^*}(U/V) \quad \text{and} \quad H^* = \bigcap_{U/V \in \mathscr{D}^*} C_{G^*}(U/V).$$

Take $U/V \in \mathscr{D}$. Then U/V is a minimal normal subgroup of G/V. By Lemma 1.1,

$$U/V \subseteq Z(F(G/V)).$$

Since $F(G^*)V/V$ is nilpotent and is also normal in G/V, we know that $F(G^*)V/V \subseteq F(G/V)$. Hence $F(G^*)V/V$ centralizes U/V. As U/V was taken arbitrarily, $F(G^*) \subseteq H$.

Clearly $H \subseteq H^*$. To complete the proof, we assume that $H^* \not\subseteq F(G^*)$ and obtain a contradiction. Let K be a normal subgroup of G minimal with respect to the property that $K \subseteq H^*$ and $K \not\subseteq F(G^*)$. Take a chief series for G that includes K, and let

$$K = K_0 \supset K_1 \supset \cdots \supset K_n = 1 \tag{1.1}$$

be the part of the chief series from K to 1. By the choice of K, we have $K_1 \subseteq F(G^*)$. Hence, for $i = 2, \ldots, n$, we have $K_{i-1}/K_i \in \mathscr{D}^*$ and, since $K \subseteq H^*$, we have $[K_{i-1}, K] \subseteq K_i$. Since K is solvable, K/K_1 is abelian and $[K_0, K] = [K, K] \subseteq K_1$. Thus the series (1.1) is a central series for K. Hence K is nilpotent. Therefore $K \subseteq F(G^*)$, a contradiction. □

Proposition 1.3 (P. Hall). Suppose that G is a solvable group. Then $C_G(F(G)) \subseteq F(G)$.

Proof. Let $G^* = G$ in Proposition 1.2. □

Proposition 1.4. Suppose that G is a solvable group, A is a group of automorphisms of G, and $(|A|, |G|) = 1$. Then A acts faithfully on $F(G)$.

Proof. We may assume that A is cyclic. Let X be the semidirect product of G by A. Then X is solvable. We embed A and G in X. Let $\sigma = \pi(A)$ and $F = F(X)$.

Since A is certainly a Hall σ-subgroup of X and $A\mathcal{O}_\sigma(F)$ is a σ-group, $A = A\mathcal{O}_\sigma(GF) \supseteq \mathcal{O}_\sigma(F)$. As $A \subseteq \operatorname{Aut} G$ and

$$[\mathcal{O}_\sigma(F), G] \subset \mathcal{O}_\sigma(F) \cap G = 1,$$

we have $\mathcal{O}_\sigma(F) = 1$. Thus

$$F = \mathcal{O}_\sigma(F) \times \mathcal{O}_{\sigma'}(F) = \mathcal{O}_{\sigma'}(F) \subseteq \mathcal{O}_{\sigma'}(X) = G.$$

Clearly $F = F(G)$. By Proposition 1.3,

$$C_A(F) = A \cap C_X(F(X)) \subseteq A \cap F(X) \subseteq A \cap G = 1. \quad \square$$

Proposition 1.5. Suppose that G is a solvable group, A is an operator group on G, and $(|A|, |G|) = 1$. Let π be a set of primes. Then:

(a) A fixes some Hall π-subgroup of G;
(b) every A-invariant π-subgroup of G is contained in an A-invariant Hall π-subgroup of G;
(c) if H_1 and H_2 are A-invariant Hall π-subgroups of G, then H_1 and H_2 are conjugate by an element of $C_G(A)$;
(d) if H is any A-invariant normal subgroup of G, then $C_{G/H}(A)$ is the image of $C_G(A)$ in G/H; and
(e) if $C_G(A)$ contains a Hall π'-subgroup of G, then $[G, A] \subseteq \mathcal{O}_\pi(G)$.

Proof. Statements (a), (c), and (d) follow from P. Hall's theorem on solvable groups (**G**, Theorem 6.4.1, p. 231) and from the proof of Theorem 6.2.2, pp. 224–5 of **G**.

To prove (b) we proceed by induction on $|G|$. Let K be an A-invariant π-subgroup of G and M a minimal A-invariant normal subgroup of G. If G itself is a π-group, there is nothing to prove, and so we may assume G is not a π-group. Now KM/M is an A-invariant π-subgroup of G/M so, by induction, there exists an A-invariant Hall π-subgroup H/M of G/M that contains KM/M. Thus H is an A-invariant subgroup of G such that $K \subseteq H \subseteq G$ and $|H|_\pi = |G|_\pi$. If $H \neq G$, we can apply induction to H to conclude that K is contained in an A-invariant Hall π-subgroup of H and we are done. If $H = G$, then M is a normal Sylow p-subgroup of G for some prime p outside π. By (a), G has an A-invariant Hall π-subgroup Q and clearly $G = QM$ with $Q \cap M = 1$. Now $|Q \cap KM| = |K|$, and hence K and $Q \cap KM$ are both A-invariant Hall π-subgroups of KM. By (c), there exists an element $x \in C_{KM}(A)$ such that $K = (Q \cap KM)^x \subseteq Q^x$. Clearly Q^x is an A-invariant Hall π-subgroup of G.

To prove (e), let H be an A-invariant Hall π-subgroup and let K be a Hall π'-subgroup of G contained in $C_G(A)$. Then $G = KH$. Therefore

$$[G, A] = \left\langle h^{-1}k^{-1}k^\alpha h^\alpha \mid k \in K,\, h \in H,\, \alpha \in A \right\rangle \subseteq H.$$

Since $[G, A] \lhd G$, we have $[G, A] \subseteq O_\pi(G)$. □

Proposition 1.6. Suppose that G is a solvable group, A is an operator group on G, and $(|A|, |G|) = 1$. Then:

(a) $G = C_G(A)[G, A] = [G, A]C_G(A)$;
(b) $[G, A, A] = [G, A]$;
(c) if $[G, A, A] = 1$, then A acts trivially on G;
(d) if G is abelian, then $G = C_G(A) \times [G, A]$; and
(e) if G is abelian and $C_G(A)$ contains every element of prime order in G, then A acts trivially on G.

Proof. For (a), let $H = [G, A]$ in Proposition 1.5(d). For (b) and (c), see the proof of **G**, Theorem 5.3.6, p. 181. For (d), see the proof of **G**, Theorem 5.2.3, p. 177. Finally, note that (e) follows from (d). □

In the following lemma we list some of the basic properties of the Frattini subgroup of a finite group.

Lemma 1.7. Suppose that G is a group and R is a p-group for some prime p. Then:

(a) if H is a subgroup of G and $G = H\Phi(G)$, then $G = H$;
(b) $R/\Phi(R)$ is elementary abelian;
(c) $\Phi(R) = 1$ if and only if R is elementary abelian; and
(d) $\Phi(R) = \langle R', x^p \mid x \in R \rangle$.

Proof. (a) **G**, Theorem 5.1.1, p. 173. (b) **G**, Theorem 5.1.3, p. 174. (c) **G**, Theorem 5.1.3, p. 174. (d) Let $S = \langle R', x^p \mid x \in R \rangle$. By (b), $S \subseteq \Phi(R)$. Since R/S is elementary abelian and $\Phi(R/S) = \Phi(R)/S$, (c) yields (d). □

Theorem 1.8 (Burnside). Suppose that A is an operator group on a p-group R and $(|A|, |R|) = 1$. Assume that A centralizes $R/\Phi(R)$. Then A centralizes R.

Proof. By Proposition 1.5(d), $R = C_R(A)\Phi(R)$. By Lemma 1.7(a), $R = C_R(A)$. (This is **G**, Theorem 5.1.4, p. 174.) □

Lemma 1.9. Suppose that π is a set of primes, G is a finite solvable π-group, and A is an operator group on G that stabilizes a normal series of G. Then $A/C_A(G)$ is a π-group.

Proof. It suffices to show that A acts trivially on G if A is a π'-group. This follows from Proposition 1.5(d) by induction on the length of the normal series. □

Proposition 1.10. Suppose that A is an operator group on a nilpotent group G and $(|A|, |G|) = 1$. Let $C = C_G(A)$. If $C_G(C) \subseteq C$, then A acts trivially on G.

Proof. Assume $C_G(C) \subseteq C$. Take $x \in N_G(C)$. For each $a \in A$ and $y \in C$, we know that $x^{-1}yx = (x^{-1}yx)^a = (x^a)^{-1}yx^a$ and $x^a x^{-1}$ centralizes y. Thus $x^a x^{-1} \in C_G(C) \subseteq C$. As x and a are arbitrary, A centralizes $N_G(C)/C$. Thus A stabilizes the normal series

$$N_G(C) \supseteq C \supseteq 1,$$

and hence, by Lemma 1.9, A acts trivially on $N_G(C)$. Thus $N_G(C) \subseteq C$. As G is nilpotent, $C = G$. Hence A acts trivially on G. □

Theorem 1.11. Suppose that p is an odd prime, G is a p-group, and A is a p'-group of operators on G that acts trivially on $\Omega_1(G)$. Then A acts trivially on G.

Proof. **G**, Theorem 5.3.10, p. 184. □

Corollary 1.12. Suppose that p is an odd prime, G is a p-group, E is an elementary abelian subgroup of G, and A is a p'-group of operators on G. Assume that A fixes every element of order p in $C_G(E)$. Then A acts trivially on G.

Proof. Let $C = C_G(A)$. Since $E \subseteq C_G(E)$, we know that $E \subseteq C$. Therefore $C_G(C) \subseteq C_G(E)$ and A fixes every element of $\Omega_1(C_G(C))$. Since p is odd, A fixes every element of $C_G(C)$ by Theorem 1.11. Consequently $C_G(C) \subseteq C$. By Proposition 1.10, A acts trivially on G. □

Theorem 1.13 (J. G. Thompson). Suppose that p is an odd prime and G is a nontrivial p-group. Then G contains a characteristic subgroup H that enjoys the following properties:

(a) $[H, G] \subseteq Z(H)$;
(b) H has nilpotence class at most two;
(c) H has exponent p; and
(d) $C_{\mathrm{Aut}\,G}(H)$ is a p-group.

Proof. This follows from Thompson's Critical Subgroup Theorem (**G**, Theorem 5.3.11, p. 185). As in Theorem 5.3.13 of **G** (p. 186), we let C be a critical subgroup of G and examine the properties of $\Omega_1(C)$. Let $H = \Omega_1(C)$. Then (b), (c), and (d) are proven in **G** (Theorem 5.3.13, p.186). Since C is a critical subgroup of G, $[G, C] \subseteq Z(C)$. Thus

$$[G, H] = [G, \Omega_1(C)] \subseteq [G, C] \cap H \subseteq Z(C) \cap H \subseteq Z(H).$$

This yields (a). □

Lemma 1.14. Suppose that p is a prime, T is a p-subgroup of a group G, and M is a normal p'-subgroup of G. Let $C = C_G(T)$ and $N = N_G(T)$. Then

$$C_{G/M}(TM/M) = CM/M \text{ and } N_{G/M}(TM/M) = NM/M.$$

Proof. Let

$$C^*/M = C_{G/M}(TM/M) \text{ and } N^*/M = N_{G/M}(TM/M).$$

Clearly $NM \subseteq N^*$. On the other hand, take $x \in N^*$. Then x normalizes TM, so T^x is a Sylow p-subgroup of TM, and there exists $y \in M$ such that $T^x = T^y$. Then xy^{-1} normalizes T. Hence

$$xy^{-1} \in N \text{ and } x = (xy^{-1})y \in NM.$$

Thus $N^* = NM$. Now $CM \subseteq C^* \subseteq N^* = NM$. Since $T \cap M = 1$, we have $C^* \cap N = C$. Hence

$$C^* = (C^* \cap N)M = CM. \quad \square$$

Proposition 1.15. Suppose that G is a solvable group and p is a prime.

(a) **(P. Hall & G. Higman, "Lemma 1.2.3")** Assume that T is a Sylow p-subgroup of $\mathcal{O}_{p',p}(G)$. Then $C_G(T) \subseteq \mathcal{O}_{p',p}(G)$.
(b) **(D. Goldschmidt)** Assume that R is a p-subgroup of G. Then $\mathcal{O}_{p'}(C_G(R)) \subseteq \mathcal{O}_{p'}(G)$.

Proof. (a) **G**, Theorem 6.3.3, p. 228. (b) By Lemma 1.14, we may assume $\mathcal{O}_{p'}(G) = 1$. Let $M = \mathcal{O}_{p'}(C_G(R))$ and $T = \mathcal{O}_p(G)$. Then $RM = R \times M$ and M is an operator group on the p-group RT. Since $C_{RT}(R)$ normalizes M,

$$[C_{RT}(R), M] \subseteq RT \cap M = 1.$$

Therefore $C_{RT}(R)$ centralizes M. Let $C = C_{RT}(M)$. Then we have $C_{RT}(C) \subseteq C_{RT}(R) \subseteq C$. By Proposition 1.10, M centralizes T. As $T = F(G)$, we know $C_G(T) \subseteq T$. Thus $M = M \cap T = 1$. $\square$

Proposition 1.16. Suppose that p is a prime, G is a p'-group, and A is a noncyclic abelian p-group of automorphisms of G. Then

$$G = \langle\, C_G(x) \mid x \in A^{\#} \,\rangle \text{ and}$$
$$G = \langle\, C_G(Y) \mid Y \subseteq A \text{ and } A/Y \text{ is cyclic} \,\rangle.$$

Proof. The first assertion is **G**, Theorem 6.2.4, p. 225. The second then follows by induction on $|G|$. $\square$

Theorem 1.17 (D. G. Higman, "Focal Subgroup Theorem"). Suppose that G is a group, p is a prime, and S is a Sylow p-subgroup of G. Then

$$S \cap G' = \langle\, x^{-1}y \mid x, y \in S \text{ and } x \text{ is conjugate to } y \text{ in } G \,\rangle.$$

Proof. **G**, Theorem 7.3.4, p. 250. $\square$

Theorem 1.18 (Burnside). Suppose that G is a group, p is a prime, and S is a Sylow p-subgroup of G. Assume that $S \subseteq Z(N_G(S))$. Then G has a normal p-complement.

Proof. Here S is abelian. Suppose x, $y \in S$ and $x^u = y$ for some $u \in G$. Then

$$S \subseteq C_G(x),\ S \subseteq C_G(y), \text{ and } S^u \subseteq C_G(x)^u = C_G(y).$$

By Sylow's Theorem, there exists $v \in C_G(y)$ such that $(S^u)^v = S$. Then

$$uv \in N_G(S) \text{ and } x^{uv} = (x^u)^v = y^v = y.$$

Since $S \subseteq Z(N_G(S))$, we know $x^{uv} = x$. Thus $x = y$ and $x^{-1}y = 1$.

By Theorem 1.17 and the argument above $S \cap G' = 1$, and hence G' is a p'-subgroup of G. Define a normal subgroup K of G by

$$K \supseteq G' \text{ and } K/G' = \mathcal{O}_{p'}(G/G').$$

It is easy to see that $KS = G$ and $K \cap S = 1$. Thus K is a normal p-complement in G. □

Corollary 1.19. Suppose that G is a group.

(a) If S is a cyclic Sylow subgroup of G, then either $S \cap G' = 1$ or $S \subseteq G'$.
(b) If G is a Z-group, then G' is a Hall subgroup of G.

Proof. Since (b) follows from (a), we will prove only (a).

Let K be a complement to S in $N_G(S)$. By Proposition 1.6(d), we know that $S = C_S(K) \times [S, K]$. Since S is cyclic, either $S = [S, K] \subseteq G'$, or $S = C_S(K) \subseteq Z(N_G(S))$. In the latter case, $S \cap G' = 1$ by Theorem 1.18. □

Theorem 1.20 (Maschke). Suppose that G is represented by linear transformations on a vector space V over a field $\mathbf{F}$. Assume that the characteristic of $\mathbf{F}$ is zero or is a prime that does not divide $|G|$.

Then V is completely reducible under G.

Proof. **G**, Theorem 3.3.2, p. 66. □

For later reference we gather together some elementary properties of p-length.

Lemma 1.21. Suppose that G is a finite group. Then:

(a) if G has p-length one and H is a subgroup of G, then H has p-length one;
(b) if H is a normal p'-subgroup of G and G/H has p-length one, then G has p-length one;
(c) if H is a normal p-subgroup of G such that $\mathcal{O}_{p'}(G/H) = 1$ and if G/H has p-length one, then G has p-length one;
(d) G has p-length one if and only if the subgroup of G generated by all p-elements of G has a normal p-complement; and
(e) if H and N are normal subgroups of G such that $H \cap N = 1$ and G/H and G/N both have p-length one, then G has p-length one.

Proof. (a), (b), and (c) are easily verified from the definition. For (d), let $U \lhd G$ be generated by all p-elements of G. If U has a normal p-complement K, then $K \operatorname{char} U$, so $K \lhd G$ and $K \subseteq O_{p'}(G)$. Clearly $UO_{p'}(G) = O_{p',p}(G)$ and $G = O_{p',p,p'}(G)$, as desired. Conversely, if $G = O_{p',p,p'}(G)$, then $U \subseteq O_{p',p}(G)$. Thus $O_{p'}(G) \cap U \lhd U$ and

$$U/(O_{p'}(G) \cap U) = UO_{p'}(G)/O_{p'}(G) \subseteq O_{p',p}(G)/O_{p'}(G),$$

which is a p-group. Thus $O_{p'}(G) \cap U$ is a normal p-complement in U.

For (e), suppose G/H and G/N have p-length one, and let U be generated by all elements of p-power order in G. Applying (d) to G/H and G/N, we can find subgroups A and B of G such that $H \subseteq A \lhd UH$ and such that $N \subseteq B \lhd UN$ and A/H and B/N are normal p-complements in UH/H and UN/N, respectively. Clearly $A \cap B \cap U$ contains all of the p'-elements of U. On the other hand, if $g \in A \cap B$ is a p-element, then $g \in H$ and $g \in N$. Thus $g = 1$ and we conclude that $A \cap B \cap U$ is a normal p-complement in U. Now (d) yields (e). □

Lemma 1.22. Suppose that p is a prime, G is a p-group, and $N \lhd G$. Let $|N| = p^k$. Then, for every nonnegative integer r such that $r \leq k$, N contains a normal subgroup of G having order p^r.

Proof. We use induction on $|G|$. The result is trivial if $N = 1$ or $r = 0$. Hence we assume that $N \neq 1$ and $r \geq 1$. Thus $N \cap Z(G) \neq 1$.

Take a subgroup Z of order p in $N \cap Z(G)$. By induction, N/Z contains a normal subgroup L/Z of G/Z having order p^{r-1}. Then $|L| = p^r$, $L \subseteq N$, and $L \lhd G$. □

2. General Results on Representations

In this section, we consider representations of groups by matrices of finite degree or by finite-dimensional linear transformations. Assume G is a group. If G acts faithfully on a vector space V over a field $\mathbf{F}$, we denote the *enveloping algebra* of G over $\mathbf{F}$ by $\mathrm{E}(G)$ (as in **G**, p. 82). This is the smallest $\mathbf{F}$-subalgebra of $\mathrm{Hom}_{\mathbf{F}}(V, V)$ that contains G. As usual, we embed $\mathbf{F}$ in $\mathrm{Hom}_{\mathbf{F}}(V, V)$ by identifying field elements with scalar multiplications.

By *module* we will always mean finite-dimensional right module.

Suppose H is a subgroup of G. If L is an $\mathbf{F}G$-module, we denote the restriction of L to H by $L|_H$ or L_H. If M is an $\mathbf{F}H$-module, we denote by M^G the $\mathbf{F}G$-module induced from M. We consider M^G to be the tensor product $M \otimes_{\mathbf{F}H} \mathbf{F}G$. If, in addition, $H \lhd G$ and $x \in G$, we denote by M^x the $\mathbf{F}H$-module with underlying $\mathbf{F}$-module M and H-action (temporarily denoted by $*$) defined by

$$m * h = m(xhx^{-1}),$$

for all $m \in M$ and $h \in H$. The module M^x is called a *conjugate* $\mathbf{F}H$-module and is clearly isomorphic to the $\mathbf{F}H$-submodule $M \otimes x$ of M^G.

Proposition 2.1. Suppose that G is a group, $\mathbf{F}$ is a field, and M is an irreducible $\mathbf{F}G$-module. Then:

(a) M is absolutely irreducible if and only if $\mathrm{Hom}_{\mathbf{F}G}(M, M) = \mathbf{F}$;
(b) if G is faithful on M and $\mathrm{Hom}_{\mathbf{F}G}(M, M) = \mathbf{F}$, then $\mathrm{Hom}_{\mathbf{F}}(M, M) = \mathrm{E}(G)$; and
(c) if $\mathbf{F}$ is a finite field and $\mathbf{K} = \mathrm{Hom}_{\mathbf{F}G}(M, M)$, then $\mathbf{K}$ is a finite field and we can regard M as an absolutely irreducible $\mathbf{K}G$-module.

Proof. (a) If $\mathbf{F}$ has characteristic zero or relatively prime to $|G|$, this is **G**, Theorem 3.5.7, p. 80. The general case can be deduced in one direction from the final paragraph of the proof in **G** (where one assumes that $D \neq \mathbf{F}$) and in the other from the Jacobson Density Theorem (**G**, Theorem 3.6.2, p. 86). For a nice proof of the general case, see [3, Theorem 29.13, p. 202].

(b) follows from the Jacobson Density Theorem (**G**, Theorem 3.6.2, p. 86) and the fact that $\mathrm{Hom}_{\mathbf{F}G}(M, M) = \mathrm{Hom}_{\mathrm{E}(G)}(M, M)$.

(c) By Schur's Lemma (**G**, Theorem 3.5.2, p. 76), $\mathbf{K}$ is a division algebra with $\mathbf{F}$ in its center. Since $\mathbf{F}$ is finite and M has finite dimension, M is actually finite. Therefore $\mathbf{K}$ is also finite and, by Wedderburn's well-known theorem on finite division rings [16, Theorem 7.2.1, p. 361], $\mathbf{K}$ is a field. Since $\mathbf{K} = \mathrm{Hom}_{\mathbf{F}G}(M, M)$, we can regard M as a vector space over $\mathbf{K}$ and the elements of G as linear transformations of M over $\mathbf{K}$. Clearly

$$\mathbf{K} \subseteq \mathrm{Hom}_{\mathbf{K}G}(M, M) \subseteq \mathrm{Hom}_{\mathbf{F}G}(M, M) = \mathbf{K}.$$

By (a), M is an absolutely irreducible $\mathbf{K}G$-module. □

Proposition 2.2. Suppose that G is a group, $H \triangleleft G$, and G/H is cyclic. Assume that $\mathbf{F}$ is an algebraically closed field and M is an irreducible $\mathbf{F}H$-module such that $M \cong M^x$ for all $x \in G$.

(a) If L is an irreducible $\mathbf{F}G$-module and M is isomorphic to a submodule of L_H, then $L_H \cong M$.
(b) The representation of H on M can be extended to a representation of G.

Proof. (a) We can assume that G acts faithfully on L. By Clifford's Theorem (**G**, Theorem 3.4.1, p. 70), there exists an integer k such that

$$L_H = M_1 \oplus M_2 \oplus \cdots \oplus M_k, \tag{2.1}$$

where $M \cong M_i$ for each i. Since G acts faithfully on L, H acts faithfully on M.

Consider, for a moment, the action of H on M. Since $\mathbf{F}$ is algebraically closed, $\mathrm{Hom}_{\mathbf{F}H}(M, M) = \mathbf{F}$ and, by Proposition 2.1, $\mathrm{E}(H) = \mathrm{Hom}_{\mathbf{F}}(M, M)$. Take $x \in G$ such that $G = \langle H, x\rangle$. Then, by hypothesis, $M \cong M^{x^{-1}}$, and therefore there is an $\mathbf{F}$-isomorphism $\tau \in \mathrm{Hom}_{\mathbf{F}}(M, M) = \mathrm{E}(H)$ such that for all $m \in M$ and $h \in H$,

$$(mh)\tau = (m\tau)xhx^{-1}. \tag{2.2}$$