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Some random series of functions

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SECOND EDITION

JEAN-PIERRE KAHANE

Professor of Mathematics, Université de Paris-Sud

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Preface

The origin of this book, whose first edition was published in 1968, is a series of notes by Paley and Zygmund that appeared in the 1930s, entitled 'On some series of functions' [167]. Paley and Zygmund, with the collaboration of Wiener in a subsequent paper, studied Fourier or Taylor series whose coefficients are independent random variables, for example

$$(\mathrm{R}): \sum_{-\infty}^{\infty} \varepsilon_n a_n \mathrm{e}^{int}, \quad (\mathrm{S}): \sum_{-\infty}^{\infty} a_n \mathrm{e}^{2\pi i \omega_n} \mathrm{e}^{int}, \quad (\mathrm{G}): \sum_{-\infty}^{\infty} a_n \zeta_n \mathrm{e}^{int}$$

where the a_n are given, and ε_n, ω_n, ζ_n stand for random variables. In (R) (Rademacher series), $\varepsilon_n = \pm 1$; in (S) (Steinhaus series), $\omega_n \in [0, 1]$; in (G) (Gaussian series), ζ_n is a complex normal variable. Steinhaus had already introduced Taylor series with coefficients $a_n \mathrm{e}^{2\pi i \omega_n}$, and Wiener had already discovered the Fourier series corresponding to brownian motion. Series (R) are sometimes more difficult to investigate than (S) or (G), but most results are the same for these three series. In the first edition of the book I took advantage of the work of Billard to explain the analogy between (R) and (S). One of the main points in the present edition is the explanation of the analogy between (R), (S) and (G); this is due to Marcus and Pisier, on the basis of the work of Dudley and Fernique on gaussian processes. Precisely, (R), (S), (G) have the same probability (0 or 1) of representing a continuous function; the same is true for an integrable function, or a function in L^p $(1 \leqslant p < \infty)$, and then the probability is 1 or 0 according to convergence or divergence of the series $\sum_1^\infty a_n^2$. The L^p-case was known in the 1930s, and the continuous case was elucidated in 1978. The analogy is not yet fully understood for Taylor series

$$(\mathrm{R}): \sum_{0}^{\infty} \varepsilon_n a_n z^n, \quad (\mathrm{S}): \sum_{0}^{\infty} a_n \mathrm{e}^{2\pi i \omega_n} z^n, \quad (\mathrm{G}): \sum_{0}^{\infty} a_n \zeta_n z^n$$

in the case $\overline{\lim}\, |a_n|^{1/n} = 1$, $\sum_0^\infty a_n^2 = \infty$. Then the range of (G) and the range of (S) fill the plane almost surely; this was proved by C. Offord in 1972 and a proof for (G) is given in this book. But the case of (R) is not yet settled.

This may give the impression of slow progress. On the other hand, random Fourier series have become a rather popular subject, with many new and important applications to harmonic analysis, described in particular in the book by Marcus and Pisier [154]. In some respects, the present book can be considered as an introduction to theirs.

Applying random methods to harmonic analysis is an old idea, and it was the first aim of my book in 1968. In many circumstances it is hard or even impossible to find a mathematical object with some prescribed properties, and pretty easy to exhibit a random object which enjoys these properties almost surely. The classical example of this situation is the following theorem: no better condition than Riesz–Fischer's, bearing on the absolute values of the coefficients of a trigonometric series, can ensure that it is a Fourier–Lebesgue series. The proof, due to Paley and Zygmund, is this: in case $\sum a_n^2 = \infty$, the corresponding Rademacher trigonometric series fails almost surely to represent an integrable function. Almost all sequences ε_n work, but no one is exhibited. Salem used random construction in order to obtain sets of multiplicity in a strong sense, which nobody knows how to construct directly, and his ideas were developed already in the first edition of the book. I now give a few other examples. Here is one: given $a_n > 0$, $\sum_0^\infty a_n^2 < \infty$, does there exist a continuous function $f \sim \sum b_n e^{int}$ such that $|b_n| \geqslant a_n$? The answer is positive and makes use of a repetition of random choices. Here is another example, too complicated to explain fully in this book: given $\varepsilon > 0$, find $N > 1$ and a polynomial $P(z) = \sum_1^N a_n z^n$, $|a_n| = 1$, such that $\sup_{|z|=1} |P(z)| \leqslant (1+\varepsilon) \inf_{|z|=1} |P(z)|$ (that is, $|P(z)|$ is almost constant on the circle $|z| = 1$). The solution came from an idea of T. Körner, that is, to apply random methods to the question.

A most beautiful example is the Pisier algebra, which solves completely an old problem of Y. Katznelson of finding a homogeneous Banach algebra of continuous functions on the circle, with the condition that not all continuous functions operate in this algebra, and not only analytic functions. There have been some difficult constructions due to M. Zafran. Pisier's solution is the algebra of continuous functions such that, changing signs randomly in their Fourier series, the resulting series represents almost surely a continuous function.

From the random point of view it often happens that the strange becomes natural (like a nowhere differentiable function, an almost everywhere divergent series, a set of multiplicity which is independent over the rationals, and so on). That proves especially true in the kind of geometrical figures that B. Mandelbrot called fractals. Sets uncovered by random intervals, images of a given set by a random process, level sets of a random function or random field, were already studied in my book in 1968, with the help of some elementary Fourier analysis. I am doing a little more in the present edition, and I devote one chapter to an introduction to the geometrical part of the book. I have replaced the chapter on covering the circle by random arcs by an exposition of the work of L. Shepp (a necessary and sufficient condition for random covering) and of the several dimensional situation (including El Helou's method). I describe gaussian processes with stationary increments as helices in a gaussian Hilbert space, and study the sample functions – mainly the sample functions of brownian motion – in more detail. The last chapters show the interplay between Hausdorff dimensions, Lipschitz or Hölder conditions, gaussian processes, images, graphs, level sets, Fourier properties of measures. As a rough idea random sets obtained in this way fill the space as much as they can, and the spectra of random measures are smooth. Salem sets look very natural from this point of view.

I left most of the material of the first edition unchanged except for minor corrections. The new sections can be found in chapters 2 (§7), 5 (§9), 6 (§§5, 6), 12 (§7), 13 (§3). Chapters 10, 11, 15, 16, 18 are new. My best advice for the reader is to proceed by random choice through the book, and discover the connexions. There are notes and references at the end. In the first edition I felt obliged to apologize for the material which was omitted. So much is omitted now that a mere list would double this preface, and I give up mentioning any omission. I addressed my thanks to a few colleagues and friends who helped me in preparing the first edition. They extend now to many more colleagues and friends. Let me just thank again Mme Josette Dumas for her constant help in preparing both editions and her help inbetween.