The algebraic theory of modular systems

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F. S. Macaulay

with a new Introduction by

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PREFACE

THE present state of our knowledge of the properties of Modular Systems is chiefly due to the fundamental theorems and processes of L. Kronecker, M. Noether, D. Hilbert, and E. Lasker, and above all to J. König's profound exposition and numerous extensions of Kronecker's theory (p. xi). König's treatise might be regarded as in some measure complete if it were admitted that a problem is finished with when its solution has been reduced to a finite number of feasible operations. If however the operations are too numerous or too involved to be carried out in practice the solution is only a theoretical one; and its importance then lies not in itself, but in the theorems with which it is associated and to which it leads. Such a theoretical solution must be regarded as a preliminary and not the final stage in the consideration of the problem.

In the following presentment of the subject Section I is devoted to the Resultant, the case of n equations being treated in a parallel manner to that of two equations; Section II contains an account of Kronecker's theory of the Resolvent, following mainly the lines of König's exposition; Section III, on general properties, is closely allied to Lasker's memoir and Dedekind's theory of Ideals; and Section IV is an extension of Lasker's results founded on the methods originated by Noether. The additions to the theory consist of one or two isolated theorems (especially §§ 50—53 and § 79 and its consequences) and the introduction of the Inverse System in Section IV.

The subject is full of pitfalls. I have pointed out some mistakes made by others, but have no doubt that I have made new ones. It may be expected that any errors will be discovered and eliminated in due course, since proofs or references are given for all major and most minor statements.

I take this opportunity of thanking the Editors for their acceptance of this tract and the Syndics of the University Press for publishing it.

F. S. MACAULAY.

LONDON,
June, 1916.

Introduction

Many of the ideas introduced in Macaulay's book on Modular Systems have developed into central concepts in the branch of mathematics known today as Commutative Algebra. His name is remembered today mostly through the term "Cohen–Macaulay ring", a notion which grew out of the unmixedness theorem in the third chapter of this book. However, it is less well known that he pioneered several other fundamental ideas, including the concept of Gorenstein ring and the use of injective modules, ideas which were not systematically developed until considerably later in this century.

In 1916, when Macaulay's book appeared, the field of Commutative Algebra had not grown to the point where it could be considered a separate branch of Mathematics, and rings were not studied for their own sake. The topics in this book had their origins instead in the problem of finding solutions to systems of polynomial equations. This problem may be considered to be a branch of Algebraic Geometry, and many of the subjects discussed here really belong to that field. While it is not always easy (or necessary) to separate the field of Commutative Algebra from the more algebraic side of Algebraic Geometry, we concentrate here on developments in Commutative Algebra, since the main new methods introduced in this book would today be considered as belonging to that field.

This introduction has several aims. We describe many of the ideas in this book, both in their own context and how they have developed since those days. We first present a summary of how the approaches differed, and then give a more detailed account of how the individual ideas have developed. Macaulay's writing is not always easy to read, in part because of the condensed style, and in part because of the differences in terminology and notation, not to mention certain conventions which are no longer in common use. For example, the word "module" is used where we would now use the word "ideal"; while "module" has a very different meaning today. We point out some of these differences in the introduction, and at the end we summarize the main differences between his terminology and that which is currently standard. We use modern terminology unless stated otherwise. A polynomial ring in n variables will be denoted $k[x_1, \ldots, x_n]$, $k[x_i]$, or R, and an ideal in this ring will be denoted M.

One obvious difference between the content of this book and that of a modern text in the subject is that the modern theory is much more abstract, while that of Macaulay involves more specific computations. In fact, one of the reasons that Macaulay's work has been referred to steadily over the years is that it contains many useful examples which are still of widespread interest.

Background: the study of polynomial rings in Macaulay's time.

The first two chapters of "The Algebraic Theory of Modular Systems" deal mostly with the question of describing solutions to sets of polynomial equations, while the last two are concerned with the finer structure of the ideals themselves. We begin by giving a picture of the subject of describing these solutions and how it fit into the development of mathematics at the time.

The ideal solution to the question of finding all solutions to a set of equations would be to list variables which can be assigned values arbitrarily and provide a set of formulas for each of the other variables in terms of the arbitrary ones. In simple cases, such as when the equations are linear, such a program can be carried out, but such complete solutions cannot be found in general. The first two chapters of this book deal with partial solutions to this problem.

While one cannot "solve" a set of equations as above, much can be said about the set of solutions. One main fact, known well before Macaulay's time, is that the set of solutions can be divided into components, each of which is "irreducible" and has a well-defined dimension. A further major step was Hilbert's Nullstellensatz [7], in which he showed that there is a correspondence between certain ideals of the ring of polynomials and sets of solutions; on the other hand, the ideal structure was finer in that different ideals may have the same set of solutions. The work of Lasker [12] on primary decomposition showed that an ideal can also be divided into components, or, more precisely, that every ideal may be represented as an intersection of primary ideals. Certain of these primary ideals correspond to the components of the set of solutions, but in general there are others, which are called imbedded components. It was Macaulay who pointed out the importance of these imbedded components and their relation to the structure of the ideal. We discuss these questions further in the section on the third chapter of the book.

The fact that the subject was still close to the problem of finding solutions to polynomial equations may explain one convention which appears strange to a modern reader. Whereas it would seem normal today to say that an ideal M is contained in an ideal N if M is a subset of N, Macaulay would say that M contains N in this case. On the other hand, if we think of the ideals as representing their sets of solutions, Macaulay's expression is reasonable. It could be argued that the modern term for his "module" is "closed subscheme of affine space" rather than "ideal in a polynomial ring", since closed subschemes correspond to ideals with the order relation reversed. However, his arguments are ideal-theoretic, and we shall refer to them as ideals.

A comparison with the modern approach to the subject.

Since Macaulay's book appeared, the field of Commutative Algebra has changed considerably. The most obvious difference between the work of Macaulay and a modern book in Commutative Algebra is that Macaulay dealt exclusively with ideals in polynomial rings, while more recently the subject is much more abstract and deals with arbitrary "Noetherian rings". For a basic modern text on the subject we refer to *Commutative Ring Theory* by Matsumura [15]; most of the results and facts which we mention here without a particular reference may be found in Matsumura's book. We also note the historical treatments of the subject by Kaplansky [10] and Bourbaki [3]. This introduction is not intended to give a complete history of the subject, but rather to point the influences of Macaulay's work, as well as some of the ways in which the subject has changed.

Although this tendency toward abstraction has resulted in greater generality, with a much wider class of rings being studied, the case of polynomial rings is still the most important example, and much of the field is devoted to it today. In fact, polynomial rings have a more central place in the subject than they did a few years ago, partly because of the recent introduction of computer methods, which are almost entirely devoted to computations in polynomial rings. We note that Macaulay's name is also commemorated in the most widely used system at present, the program *Macaulay* of Bayer and Stillman; while the contents of this book are only indirectly related to modern computational methods, his later work (Macaulay [13]) has influenced this development.

In addition to the tendency toward abstraction, there have been several more specific changes which have served to clarify the main ideas of Commutative Algebra and to simplify many of the proofs. First was the introduction of the ascending chain condition on ideals, following the proof of Emmy Noether [16] of the existence of primary decomposition, greatly clarifying this theorem. Second was localization; in a modern treatment many, if not most, of the questions are reduced to the local case and many problems are simplified. In many places in this book, substitutes for localization are obtained through ad hoc methods. Finally, the concept of dimension, due to Krull [11], has been refined; it is no longer based on the complicated notions arising from explicit solutions to equations required in this book. We shall return these topics at greater length in further sections.

Chapters 1 and 2: The Resultant and The Resolvent.

As stated above, the first two chapters of the book are devoted to the questions of finding solutions to systems of polynomial equations, and much of the material here comes from earlier authors. Many of these topics have not been considered as essential to the subject recently as they were then, and for that reason Macaulay's book remains a good reference for this material.

The first chapter is devoted to the resultant of a set of n homogeneous polynomials in n variables. The aim of this technique is to find a polynomial in the coefficients of the n polynomials which vanishes if and only if the set of polynomials has a non-zero solution. This generalizes the well-known case of n linear homogeneous equations in n unknowns, which have a non-zero solution if and only if the determinant of the coefficient matrix vanishes. In the case of two polynomials in two variables, the resultant is again the determinant of a matrix whose entries are the coefficients; this case is still widely used today, particularly in the version of the resultant of two non-homogeneous polynomials in one variable, where it gives a criterion for the existence of a common solution to the polynomials. For polynomials in three or more variables, the resultant is defined as the greatest common divisor of a set of determinants, and is considerably more complicated. Macaulay gives a better method for constructing the resultant (§§6–7) and proves its basic properties (§§8–11), such as irreducibility and the fact that it does characterize sets of polynomials with non-zero solutions.

While most of this chapter studies the resultant for its own sake, we mention two connections with the material in later chapters. First, the resultant is defined via a matrix whose columns are indexed by the monomials in the polynomial ring and whose rows are indexed by monomial multiples of the n polynomials, so that the entries are either zeros or coefficients of the polynomials (see §1). This representation of elements of an ideal by an array of coefficients is used systematically in the fourth chapter to study more general ideals. Second, the resultant is referred to later (§67) in the discussion of multiplicities.

If the aim of the theory of the resultant is to characterize the existence of non-trivial solutions, the purpose of the resolvent is to find the solutions. In theory, the method succeeds; in practice, the computations are formidable, and they assume that every polynomial in one variable over any coefficient field can be solved. However, the resolvent, or rather the variation called the u-resolvent, is used in an important theoretical way in later chapters, so we describe this method briefly.

The basic idea here is to express the polynomials as polynomials in one variable x_n whose coefficients are polynomials in the $n-1$ variables $x_1, \ldots, x_{n-1}$, and to reduce the problem to one in fewer variables. We suppose first that the polynomials have a non-trivial common factor $f(x_i)$. In this case, the first $n-1$ variables can be chosen arbitrarily, and solving the polynomial $f(x_i)$ for x_n provides a solution to the complete set of equations. Since there are $n-1$ independent variables, the dimension of the space of solutions is $n-1$. The factor $f(x_i)$ is then factored out, and the resultant is then used to give a condition on the coefficients of the resulting polynomials (polynomials in $n-1$ variables) to have a solution, giving a new set of polynomials in one fewer variable (§§13–14). This process is continued, resulting in a (possibly trivial) factor at each stage. This factor is a polynomial in $n-k$ variables at the k^{th} stage, and is constructed in the same way as the polynomial $f(x_i)$ of the first step. The product of these factors is the total resolvent.

Before discussing how Macaulay uses this construction we make one comment on the modern way of reducing the number of variables. Now one would take the intersection of the ideal with $k[x_1, \ldots, x_{n-1}]$ via standard bases rather than use the resultant, since this requires much less computation. While determinants are of important theoretical value, they are not efficient to compute, so other methods are used.

Each factor of the resolvent gives a polynomial whose solutions have a given dimension. In each factor, certain variables are taken to be arbitrary, and the equation can be solved for the other variables. Thus the factors give the irreducible components of the set of solutions of the polynomials together with a very concrete description of their dimension.

It is in this chapter (§17) that Macaulay introduces the discussion of unmixedness, which is among his most important contributions to the field. As stated above, the factors of the resolvent determine the irreducible components of the set of solutions. Macaulay calls this set of solutions the "spread" of the ideal. The question is whether the other factors, whose solutions are properly contained in one of the components, give a meaningful description of the finer structure of the ideal and a reasonable criterion for the ideal to be "mixed". Such a possibility had been suggested by earlier authors. Through a succession of examples, Macaulay shows that the factors of the resolvent are inadequate for determining whether an ideal is mixed or not. This discussion is taken up again in the third chapter.

The concept of dimension.

Macaulay's notion of dimension of a quotient R/M, where M is an ideal, is based on the "u-resolvent" of the ideal, which is a variation on the resolvent as described above. We refer to (§§18–22) for the definitions. The u-resolvent gives the solutions in a somewhat more precise form, and, like the ordinary resolvent, gives a very concrete interpretation of the components and the dimension of the set of solutions in the number of arbitrary variables. Macaulay's use of dimension is entirely based on this construction, and he uses the irreducible components defined in this way systematically in place of the corresponding prime ideals.

Since the work of Krull [11], it has been the prime ideals themselves which form the basis for dimension theory in Noetherian rings. The modern view is much simpler, in that it is not based on the resolvent, which is by any standards a complicated construction. The stronger results of dimension theory which apply to polynomial rings but not to general Noetherian rings are based on the Hilbert Nullstellensatz [7] rather than the resolvent. We note that Macaulay never mentions Hilbert's Nullstellensatz, although he does prove and use the Hilbert basis theorem in the third chapter.

If the introduction of Krull dimension made it possible to avoid the resolvent in dimension theory, the introduction of chain conditions by Emmy Noether [16] made the arguments using induction on dimension even simpler. A striking example is the proof of "Lasker's Theorem" (now attributed to Lasker and Noether) on primary decomposition (§39 in Chapter three). This proof is also based on the *u*-resolvent, which makes the induction part much more difficult than it needs to be; this step would now be obtained by simply taking a maximal counterexample and working from there. For the modern reader, these arguments can be avoided by using the modern version of dimension and chain conditions. This dimension theory is also at the basis of Macaulay's use of prime ideals, where, as mentioned above, he usually uses the corresponding set of solutions and its representation by the "*u*-resolvent" in place of the prime ideal itself. Replacing "irreducible spread" by "prime ideal" usually gives the same result and sometimes avoids complexities which are not totally necessary.

However, while the theory of the resolvent is no longer needed as a foundation for the theory of dimension and irreducible components, these sections of the book are still very interesting in their original purpose of describing solutions to sets of polynomial equations.

Chapter 3: General Properties of Modules.

The third chapter contains the results for which Macaulay is best known, including the definition of unmixed ideals and the Unmixedness Theorem for ideals in polynomial rings. It also contains many related theorems on unmixed ideals, as well as many other theorems which make up the basic theory of Commutative Algebra, including for example the existence of primary decomposition (discussed in the previous section) and the Hilbert basis theorem.

The first several sections of the chapter (§§23–28) are devoted to the basic constructions on ideals, and are essentially the same as would appear in a modern text except for notation. His notation is based on that of elementary number theory, while more recently the convention has been to use set-theoretic notation. For instance, he will refer to the LCM (least common multiple) of ideals where now we would call it the intersection, and he writes (M/M') for $M : M' = \{m | mM' \subseteq M\}$.

In the earlier chapters he had pointed out that the method of the resolvent was inadequate in differentiating between mixed and unmixed ideals. While this method gave a satisfactory theory of the components of the set of solutions corresponding to an ideal of polynomials, when there were imbedded components it may not detect them, while it produced extraneous components where they should not be any. The method which he uses to give a good definition of unmixedness is based on Lasker's theory of primary decomposition.

This theorem states that every ideal can be written as a finite intersection of primary ideals, and that the set of solutions corresponding to a primary ideal is irreducible, and thus corresponds to a unique prime ideal. The definitions of prime and primary ideals have not changed over the years, but, as mentioned above, Macaulay consistently refers to the "irreducible spread", or set of solutions, corresponding to a primary ideal rather than to the prime ideal itself. If the representation of an ideal M as an intersection of primary ideals is made irredundant, the set of prime ideals thus obtained is unique; he calls these the "relevant spreads" of the ideal M; they would now be called the associated prime ideals of R/M.

Macaulay's main contribution to this theory was to point out the importance of what he called 'imbedded spreads", now called imbedded primes of R/M. That is, among the relevant spreads of a module, it is possible that some are properly contained in others, and this will have consequences, particularly involving zero-divisors on the quotient. He now gives the correct definition of an unmixed ideal —it is one all of whose relevant spreads have the same dimension. In particular, an unmixed ideal can have no imbedded prime ideals.

We discuss this further below, but we wish to mention one idea which is not usually included among these theorems today. In the midst of the section on prime ideals, Macaulay introduces "H-modules", which means homogeneous ideals. For a given ideal, he then defines the concept of H-basis; this means a set of generators for the ideal whose leading coefficients generate the ideal of all leading coefficients of elements of the ideal. This concept is used in the more detailed theory of ideals in the fourth chapter. Its similarity to the concept of standard (Gröbner) basis, which has become important for modern computational methods, is quite evident. Standard bases have the same property as Macaulay's H-bases, but with respect to a total order on all monomials rather than the order by degree of the polynomials. In a later work ([13]) Macaulay studied these orders on monomials in studying Hilbert polynomials of ideals, and this paper has influenced modern developments in computational Commutative Algebra.

The Unmixedness Theorem and the concept of depth.

The main theorem of this chapter, and perhaps of the entire book, is the theorem that an ideal of height r generated by r elements is unmixed

(§§44–48). He calls such an ideal a "module of the principal class"; it would now usually be called a complete intersection ideal.

This theorem, in addition to being of great interest in itself, has had a tremendous influence on later developments in the subject. The concept of "depth" originates from this theorem; to explain this connection, it is useful to examine briefly Macaulay's proof of the theorem . Starting with an ideal of height r generated by r elements, he assumes that it has an associated prime ("relevant spread") of height greater than r; such a situation would violate the theorem. He then shows that one can find a polynomial of the form $x_i - a_i$ (perhaps after a change of co-ordinates) so that the new ideal generated by $r+1$ elements has an associated prime of height at least $r+2$. Continuing in this way, he eventually produces a complete intersection ideal of height less than n with a maximal associated prime ideal. In a separate Lemma (§44), he shows that this situation is impossible.

We now compare this proof to the modern idea of depth. The depth of a local ring can be defined as follows: if the maximal ideal is an associated prime ideal, the depth of the ring is zero. If not, there exists a non-zero-divisor in the maximal ideal of the ring, and one can divide by it. One can continue dividing by non-zero divisors until the maximal ideal is an associated prime ideal of the quotient. The number of steps, or the total number of successive non-zero-divisors found, is defined to be the depth of the ring. Such a sequence of non-zero-divisors is called a *regular sequence*. Thus Macaulay showed that the depth equals the dimension for a ring of the type R/M where M is generated by the number of elements equal to its height, and in the process he showed that if the depth is equal to the dimension, the ideal is unmixed.

This theorem was generalized to the case of a regular local ring some years later by Cohen [4]. Since then both of their names have been associated to this theorem, and rings which satisfy the theorem that an ideal of height r generated by r elements is unmixed have been called "Cohen–Macaulay rings". While Macaulay does not quite define this property, he does give an example to show that an unmixed ideal M does not necessarily have the property that $(M, x_i - a_i)$ is unmixed; this example is now the standard example of a non-Cohen–Macaulay integral domain. (§44).

While this concept has been used for some time to prove unmixedness of certain ideals, it was the introduction of homological algebra which led to an understanding of the true importance of this concept. For one thing, there are several equivalent definitions of depth based on the vanishing of homology modules (see Matsumura [15], chapter 6, for example) which make the theory simpler. Second, the Auslander–Buchsbaum theorem [1] gives an extremely useful relation between homological dimension and depth. This theorem states that if T is a module of finite projective dimension, we have

$$\text{depth}(T) + \text{proj.dim.}(T) = \text{depth}(R).$$

In addition, there are several vanishing theorems, such as Peskine and Szpiro's "Lemme d'acyclicité" [17] which say that if the depth of the ring is large enough, and the dimension of the homology of a complex is small enough, the complex is exact. These theorems have become the standard method for proving that a complex constructed as a free resolution of a module is in fact a resolution. As an example, the exactness of a resolution of an ideal generated by determinants is usually established in this way. While Macaulay never considers free resolutions in the modern sense, he has given an idea of this method in §§52 and 53, where he shows that the relations between the determinants are the predicted ones if the height of the ideal they generate is large enough. In addition, he shows that the ideal is unmixed in this case; these arguments may be considered to be precursors of the modern homological methods.

The concept of depth and the introduction of homological algebra have made it possible to prove many theorems relating depth, homological dimension, and multiplicities in the case of Cohen–Macaulay rings (while we have not yet mentioned multiplicities, they are discussed in the fourth chapter of the book). One of the main developments in the field of Commutative Algebra in the last twenty years or so has been has been to prove these in the case in which they were not Cohen–Macaulay, as well as many related conjectures. A major step was the introduction of the Frobenius map and reduction of these questions to positive characteristic by Peskine and Szpiro [17]. These techniques were further developed by Hochster [8], where a thorough discussion and list of these conjectures may be found. Recent progress on them is described in Roberts [19].

Chapter 4: The Inverse System.

The last chapter of the book is devoted to the "inverse system" of an ideal, which is one of the most original ideas in the book and one which Macaulay himself describes as "new". The basic idea of the inverse system is to study an ideal by investigating its dual, where the dual of an ideal is meant in a sense which will be described more precisely below. Using this idea, Macaulay studies deeper properties of ideals, including the concept of a principal system, which would now be called a Gorenstein ideal, and which has become another of the fundamental concepts in Commutative Algebra. In addition, it has led to very general duality theorems in Algebraic Geometry. Among the theorems in this chapter, he proves that a complete intersection is Gorenstein.

The construction of the inverse system.

The basic construction is fairly simple. In modern terms, it would be described as follows: corresponding to an ideal M in a polynomial ring $k[x_i]$, the inverse system is the dual of $k[x_i]/M$ over k, or $\mathrm{Hom}_k(k[x_i]/M, k)$.

Macaulay's actual construction of the inverse system is considerably more concrete (§57). If M is an ideal of $k[x_i]$, the dual of $k[x_i]/M$ is naturally imbedded in the dual of $k[x_i]$. The dual of $k[x_i]$ is identified with a power series ring $k[[x_i^{-1}]]$. The monomials in x_i^{-1} are then dual to the corresponding monomials in x_i, and the natural structure of $k[x_i]$-module is ordinary multiplication, where we let any monomial in the product with at least one positive exponent equal 0. The duality pairing can be recovered as follows: if $f(x_i)$ is a polynomial and $g(x_i^{-1})$ an element of the inverse system, the value of $g(x_i^{-1})$ paired with $f(x_i)$ is the constant term of $f(x_i)g(x_i^{-1})$. The inverse system of M is then identified with the set of all power series which, when paired with all elements of M, give zero. If $g(x_i^{-1})$ is such an element, the equation $g(x_i^{-1}) = 0$ is called a "modular equation" of M.

As an example of this notation, which is used throughout the chapter, we note the rather disconcerting expression "$1 = 0$", which occurs several times. This equation should be interpreted to mean that "$1 = 0$" is a modular equation of M, which in turn means that the constant term of $f(x_i) \cdot 1$ is zero for all $f(x_i) \in M$. Thus, it means simply that M is contained in the ideal generated by $x_1, \dots, x_n$.

As mentioned above, his construction of the inverse system of an ideal is very concrete, and it is based on a picture of the ideal which he calls the "dialytic array" (§59). The dialytic array corresponding to the ideal M is a list of all elements in a basis for M as a vector space over k arranged by degree. There are several versions of this construction, but in all of them one obtains an infinite matrix with rows indexed by elements of M and columns by monomials; this matrix is analogous to the matrix used to define the resultant in Chapter 1. The inverse system is then represented in a similar way with entries dual to those in the dialytic array of M.

We remark on one other convention which could be interpreted differently today, and that is that operations such as LCM defined on inverse systems mean the corresponding operation (in this case intersection) on the corresponding ideals, and not the intersection of the submodules of $k[x_i^{-1}]$, which could be inferred by a modern reader.

This construction has been transformed considerably over the years. In a modern context, one would consider a local ring A with residue field k, and the present version of the inverse system is the injective hull of the residue field of k, denoted $E(k)$. If one applies this to the localization of a polynomial ring at the maximal ideal generated by $x_1, \ldots, x_n$, we get a construction almost identical to that of Macaulay, but with polynomials in x_i^{-1} rather than arbitrary power series. It is interesting to note that Macaulay does single out the elements of the dual which have a finite number of non-zero coefficients, which he calls "Noetherian equations", and in effect uses this as a substitute for localization at the origin (§65). If M is an ideal of A, the submodule $\mathrm{Hom}(A/M, E(k))$ is the analogue of Macaulay's inverse system, and is usually called the Matlis dual of A/M; this terminology has been extended to A-modules, where it forms the basis for a duality theory between Noetherian and Artinian modules over a complete local ring (see Matlis [14]).

Uses of the Inverse System.

Next, we discuss how Macaulay applied the inverse system to the study of ideals. There are two parts to this discussion, the first dealing with ideals with quotients of finite length, and the second to more general unmixed ideals. The most significant property defined here is undoubtedly that of a principal system, which means an ideal whose dual is generated by one element. In the dimension zero case, this condition on an ideal M is precisely the condition for R/M to be a Gorenstein ring.

Macaulay first studies in detail the case in which $k[x_i]/M$ is a finite length ring supported at the origin, in which case he calls M a "simple module". (§ 67–75). The major result here is that an ideal of the "principal Noetherian class" (that is, an ideal whose localization at the origin is a complete intersection) is a principal system. In modern terms, this is the fundamental theorem that a local complete intersection is Gorenstein. He also gives an example of a Gorenstein ring which is not a complete intersection. He then proves that there is a unique element of R/M annihilated by the maximal ideal and that in the homogeneous case, if this element has degree d, there is a duality between elements of R/M of degree m and those of degree $d-m$. His proof assumes that the coefficient field is the field of real numbers, so that one can obtain dual elements which do not give zero when paired with the given element by requiring the corresponding coefficients in the dual basis to be equal to the original ones, so that the pairing will result in a sum of squares, which cannot be zero. However, this argument can be modified to give a proof which works in general. In addition to the algebraic generalizations we have been discussing, this theorem, together with the duality, has been used in the theory of residues in Algebraic Geometry, where it is often referred to as "Macaulay's Theorem", see Griffiths [5].

Generalizations of the concept of principal system have been as far-reaching and as basic to the subject as have those of depth and Cohen–Macaulay rings, and, as in that case, the subject took on new development with the advent of homological algebra. First, it has been generalized to arbitrary dimension in somewhat the same way that regular sequences have been, and is, as always, considered over arbitrary local rings. The main impetus to the development of this topic was the paper of Bass [2], where he generalized the condition to arbitrary dimension and showed that it was equivalent to the property that the ring has finite injective dimension. In the zero dimensional case, this says that the ring is its own injective hull, and the duality properties proven by Macaulay are then consequences of the injectivity of R/M as an R/M-module.

We mention one further development in this direction. We have already pointed out the generalization of the inverse system to the Matlis dual. This theory has also been generalized using more advanced methods of homological algebra to give a duality in the derived category of bounded complexes of modules with finitely generated homology, which is callled Grothendieck duality (see Hartshorne [6]). This theory has been extended to all subschemes of regular schemes, and Gorenstein schemes are characterized as those for which the dualizing complex is a locally free module, so locally generated by one element. While this is far from Macaulay's principal systems, it can be seen to be a direct analogue of Macaulay's condition that the inverse system be generated by one element.

Multiplicities.

While this subject is not treated extensively in this book, it is dealt with briefly and has become important later. The multiplicity of an ideal M such that R/M has finite length is the length of the localization of R/M at the maximal ideal $(x_1,\dots,x_n)$. Macaulay defines the multiplicity using the inverse system, because in this way it can be defined as the dimension of a susbspace of the dual, avoiding the necessity of localizing the quotient module at the origin. The main theorem is that it the multiplicity of R/M agrees with that given by the resultant in the case in which M is a complete intersection. In addition, he proves some results on multiplicities for ideals which are residual with respect to a Gorenstein ideal (§§73–76). He states that the multiplicity has "no geometric significance" for ideals which are not complete intersections; however, this concept has since been generalized in several ways both in algebraic and geometric settings and its geometric significance for more general ideals is now well-established (see Samuel [20] and Serre [21]).

Perfect ideals.

After the discussion of primary modules, Macaulay takes up the subject of ideals of height less than the dimension of the ring. The main tool is the dialytic array mentioned above, and the main concept is that of "perfect module". He also considered "mutually residual modules".

While the computations in this section are quite complicated, we describe a part of the construction briefly. In considering a height r ideal M, usually assumed to be unmixed, Macaulay looks at it as an ideal in variables $x_1, \dots x_r$ with coefficients in $k(x_{r+1}, \dots, x_n)$, thus effectively localizing to reduce to the case where the quotient has finite length. This ideal is denoted $M^{(r)}$. Since the quotient now has finite length, there are a finite number of monomials such that every monomial can be written as a linear combination of these modulo $M^{(r)}$. However, he then examines the dyalitic array in detail and examines which elements of $k[x_{r+1}, \dots, x_n]$ must be inverted to solve for the remaining monomials in terms of the basis of the quotient. The element which must be inverted to solve the equations is denoted R (§79). In particular, he describes the inverse system of $M^{(r)}$ quite explicitly. If nothing has to be inverted, so that $R = 1$, he calls the ideal perfect.

We recall that a module T is perfect in the modern sense if it has finite projective dimension and the projective dimension is equal to the length of a maximal regular sequence in its annihilator. For regular rings, using the connection between projective dimension and depth and the fact that every module has finite projective dimension, this condition is equivalent to the condition that T be Cohen–Macaulay. Thus for an ideal M in a polynomial ring, R/M is perfect if and only if R/M is Cohen–Macaulay. For general ideals, Macaulay's condition is stronger than the modern notion; for example he gives an example of an ideal generated by a regular sequence which is not perfect in his sense. However, for homogeneous ideals the two notions are equivalent. It is quite remarkable that he comes to the same idea by such a different route, and it is an interesting exercise to trace the connection between zero divisors in a system of parameters as in the modern definition and "extra rows" in the dialytic array as in Macaulay's (§77).

Macaulay also states the theorem that, if M is a homogeneous ideal and the quotient R/M is divided by a linear system of parameters, the multiplicity of the result gives the length of $R/M^{(r)}$ if and only if the ideal is perfect, and otherwise it is larger (§89). He does not prove this theorem here, as he evidently considered it to be obvious from his construction (a proof of a modern version can be found in Matsumura [15], Theorem 17.11). The idea here is that the length of the result of dividing by a system of parameters is the correct result if and only if the ring is Cohen–Macaulay. Much has been written on the problem of defining the multiplicity in such a way as to give the correct result in general since then. There have been several solutions to this question, of which we mention two. One approach is to define the multiplicity as a limit over powers of the ideal generated by the parameters as in Samuel [20]. A different method is to use the Euler characteristic of a Koszul complex on the parameters as in Serre [21]. Both methods give the same result, and both give Macaulay's answer in the case where R/M is Cohen–Macaulay.

We remark finally that near the end of the chapter there are two sections (§§86–87) on ideals which are mutually residual with respect to a complete intersection ideal. This is another subject which he treats rather briefly but which has had a great development in recent years. It is now called either liaison or linkage, and its present development started with the paper on liaison by Peskine and Szpiro [18]. Since then many other papers have been written on properties of linked ideals; for an account of more recent work on the subject we refer to the paper of Huneke and Ulrich [9].

Summary of Macaulay's Terminology.

In this section we summarize the main differences between the terminology used in Macaulay's book and that which is used today. We have not attempted to include those terms which do not have modern equivalents, such as "dialytic array" which are defined in the book.

Some of these terms have already been discussed in earlier sections. In particular, we have mentioned his use of the word "module" where today we would say "ideal". In the following list M and N denote ideals in a polynomial ring. We also recall that he used the term "spread" of an ideal to denote its set of solutions in affine space, and that he used the irreducible spread in place of the prime ideal in many cases. While these are not strictly speaking the same, we have used their equivalence to identify them in this list. Finally, we say "Cohen–Macaulay ideal" I to denote one for which $k[x_i]/I$ has this property; this is often used today. Also, in some cases the modern term has a more general meaning, and they are equivalent only in the cases considered by Macaulay.

Term used by Macaulay	*Modern term*
module	ideal
G.C.M of $M_1, \ldots, M_k$	$M_1 + \ldots + M_k$
L.C.M. of $M_1, \ldots, M_k$	$M_1 \cap \ldots \cap M_k$
M/N	$M : N$
M contains N	M is contained in N
dimensions	dimension
rank of M	height of M
simple module M	ideal M such that R/M is Artinian
H-module	homogeneous ideal
Noetherian module	ideal all of whose associated prime ideals are contained in $(x_1, \ldots x_n)$
relevant spread	associated prime ideal
spread of M	set of solutions of polynomials in Min k^n
module of the principal class	complete intersection ideal
principal system	Gorenstein ideal
perfect H-module	homogeneous ideal such that R/M is perfect
inverse system	Matlis dual
A-derivate of E (for E in the inverse system)	product of A and E

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