I

Functions Harmonic in $|z| < 1$. Rudiments

A. Power series representation

Let $U(z)$ be real and harmonic in $|z| < R$. (This means $U(z)$ is infinitely differentiable there and satisfies

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0.$$

We write throughout $z = x + iy$.) Then we can construct another real function $V(z)$, harmonic in $|z| < R$, such that

$$F(z) = U(z) + iV(z)$$

is analytic there. This function V is frequently called a harmonic conjugate of U. The construction of V is completely elementary; one way of doing it as follows:

We want a function V, infinitely differentiable in $|z| < R$ which, with U, will satisfy the Cauchy–Riemann equations*

$$\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$$

$$\frac{\partial V}{\partial y} = \frac{\partial U}{\partial x}.$$

(Then we will automatically have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.)$$

Such a function V can be found by second year calculus if the differential

$$\frac{\partial U}{\partial y}\,\mathrm{d}x - \frac{\partial U}{\partial x}\,\mathrm{d}y$$

* E. Trubowitz observed that these were incorrectly written in the first edition!

is exact in $|z| < R$. But it *is* since

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0\ !$$

Again by second year calculus, any two functions V which we can find will differ by a constant. Frequently the constant is chosen so as to make $V(0) = 0$.

Once V is found, we have, for $|z| < R$,

$$U(z) = \Re F(z)$$

with $F(z) = \sum_0^\infty a_n z^n$, the power series expansion being uniformly convergent on compact subsets of $|z| < R$. That's because any function analytic in $|z| < R$ has such a power series development.

Writing $z = re^{i\theta}$, we easily find

$$U(re^{i\theta}) = \sum_{-\infty}^{\infty} A_n r^{|n|} e^{in\theta},$$

with

$$\begin{cases} A_n = \frac{1}{2} a_n, & n > 0 \\ A_0 = \Re a_0 \\ A_n = \frac{1}{2}\bar{a}_{-n}, & n < 0. \end{cases}$$

Thus, *any function $U(z)$ harmonic in $|z| < R$ has a series representation*

$$U(re^{i\theta}) = \sum_{-\infty}^{\infty} A_n r^{|n|} e^{in\theta}$$

uniformly convergent on compact subsets of $|z| < R$.

B. Poisson's formula

The formula derived in the last section can be put in closed form. If $R > 1$ we easily find, for $r < 1$,

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} U(e^{it}) \sum_{-\infty}^{\infty} r^{|n|} e^{in(\theta - t)}\, dt.$$

By summing two geometric series we get

$$\sum_{-\infty}^{\infty} r^{|n|} e^{in\phi} = \frac{1 - r^2}{1 + r^2 - 2r\cos\phi} \quad \text{if } 0 \le r < 1.$$

Thus we have derived *Poisson's representation*: *If $U(z)$ is harmonic for $|z| < R$, if $R > 1$, and if $0 \le r < 1$,*

$$U(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2) U(e^{it})}{1 + r^2 - 2r\cos(\theta - t)}\, dt.$$

This formula is basic for the whole course – we shall soon see that it holds under much more general conditions than the one stated above. We call

$$P_r(\theta) = \frac{1-r^2}{1+r^2-2r\cos\theta}$$

the *Poisson kernel* for $|z| < 1$.

C. Poisson representation of harmonic functions in various classes

Suppose we merely know that $U(z)$ is harmonic in $|z| < 1$. It is remarkable that some version of the Poisson representation will frequently hold for U in that circle.

Theorem *Let $p > 1$, let $U(z)$ be harmonic in $|z| < 1$, and suppose the means*

$$\int_{-\pi}^{\pi} |U(re^{i\theta})|^p \, d\theta$$

are bounded for $r < 1$. Then there is an $F \in L_p(-\pi, \pi)$ with

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-t)} F(t)\, dt$$

for $r < 1$.

Proof For $p > 1$, L_p is the dual of L_q, where $1/p + 1/q = 1$. The functions

$$U_n(\theta) = U\left(\left(1-\frac{1}{n}\right)e^{i\theta}\right)$$

(instead of $1 - 1/n$, any sequence of r_n tending to 1 from below will do!) have $||U_n||_p \le C$ ($||\ ||_p$ is here taken over $[-\pi, \pi]$, of course!), so, by the Cantor diagonal process, we can extract a subsequence U_{n_j} of them such that

$$LG = \lim_{j\to\infty} \int_{-\pi}^{\pi} G(\theta) U_{n_j}(\theta)\, d\theta$$

exists for G ranging over a *countable dense subset* of L_q. Since $\left|\left|U_{n_j}\right|\right|_p \le C$, this limit, LG, will actually exist for *all* $G \in L_q$ (easy exercise), and LG is then a bounded linear functional on L_q. So, since L_p is the dual of L_q, there *is* an $F \in L_p$ with

$$LG = \int_{-\pi}^{\pi} F(\theta)G(\theta)\, d\theta$$

for all $G \in L_q$.

Now for each n,

$$u_n(z) = U\left(\left(1-\frac{1}{n}\right)z\right)$$

is harmonic for

$$|z| < \frac{1}{1-(1/n)},$$

so, if $r < 1$,

$$u_{n_j}(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)u_{n_j}(e^{it})\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)U_{n_j}(t)\,\mathrm{d}t\,.$$

Fix any $r < 1$ and any θ, and use $G(t) = P_r(\theta - t)$; $G \in L_q$. Then

$$\lim_{j\to\infty}\int_{-\pi}^{\pi} P_r(\theta - t)U_{n_j}(t)\,\mathrm{d}t = LG = \int_{-\pi}^{\pi} G(t)F(t)\,\mathrm{d}t = \int_{-\pi}^{\pi} P_r(\theta - t)F(t)\,\mathrm{d}t.$$

The leftmost member is

$$\lim_{j\to\infty} 2\pi u_{n_j}(re^{i\theta}) = 2\pi U(re^{i\theta}).$$

Thus,

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)F(t)\,\mathrm{d}t,$$

where $F \in L_p$. Q.E.D.

Remark The same result holds, with the same proof, for $p = \infty$, if we change the statement slightly:

Theorem *If $U(z)$ is harmonic and bounded in $|z| < 1$, there is an $F \in L_\infty(-\pi, \pi)$ with*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)}F(t)\,\mathrm{d}t.$$

If I am not mistaken, this result was proved by Fatou, in his famous thesis, *Séries trigonométriques et séries de Taylor*, published before the First World War. The result is indeed the *starting point* of the whole subject treated here. Many of the ideas in the first half of this book have their origin in Fatou's thesis.

What if $p = 1$? $L_1(-\pi, \pi)$ is, unfortunately not the dual of *anything*. But M – the space of finite signed measures μ on $[-\pi, \pi]$ – with $||\mu||$ = total variation of μ – *is* the dual of $\mathscr{C}[-\pi, \pi]$ – the space of continuous functions on $[-\pi, \pi]$. If $p \in L_1[-\pi, \pi]$, we can associate to p a signed measure μ_p by putting

$$\int_{-\pi}^{\pi} G(t)\,\mathrm{d}\mu_p(t) = \int_{-\pi}^{\pi} G(t)p(t)\,\mathrm{d}t;$$

then $||\mu_p|| = ||p||_1$.

Here, then, the argument used in proving the first theorem of this section gives:

Theorem *If $U(z)$ is harmonic in $|z| < 1$ and the means*

$$\int_{-\pi}^{\pi} |U(re^{i\theta})|\,\mathrm{d}\theta$$

are bounded for $r < 1$, there is a finite signed measure μ on $[-\pi, \pi]$ with

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\,\mathrm{d}\mu(t), \quad 0 \le r < 1.$$

Corollary (Evans) *Let $U(z)$ be harmonic in $|z| < 1$ and* positive *there (here and henceforth 'positive' just means 'non-negative'). Then there is a finite* positive *measure μ on $[-\pi, \pi]$ with*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} P_r(\theta - t)\,\mathrm{d}\mu(t), \quad 0 \le r < 1.$$

Proof For $r < 1$ (using, e.g., the expansion

$$U(re^{i\theta}) = \sum_{-\infty}^{\infty} a_n r^{|n|} e^{in\theta}$$

valid in $|z| < 1$), we have

$$2\pi U(0) = \int_{-\pi}^{\pi} U(re^{i\theta})\,\mathrm{d}\theta = \int_{-\pi}^{\pi} |U(re^{i\theta})|\,\mathrm{d}\theta,$$

since $U \ge 0$. Now just apply the theorem. The measure μ is positive because here (look again at the proof of the first theorem in this section)

$$\int_{-\pi}^{\pi} G(t)\,\mathrm{d}\mu(t)$$

comes out *positive* for each *positive* $G \in \mathscr{C}$ – it's the *limit* of positive things!

D. Boundary behaviour

If we have one of the representations

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)} F(t)\,\mathrm{d}t$$

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\theta - t)}\,\mathrm{d}\mu(t)$$

derived in the previous section, we should examine the connection between $U(z)$ and the function $F(t)$ or the measure $\mathrm{d}\mu(t)$.

1. Integrability properties; functions given by Poisson's formula

We first obtain some crude results which are sufficient for many investigations.

The Poisson kernel

$$P_r(\phi) = \frac{1 - r^2}{1 + r^2 - 2r\cos\phi} = \sum_{-\infty}^{\infty} r^{|n|} e^{in\phi}$$

has the following properties:

(a) $P_r(\phi) > 0, \quad r < 1$
(b) $P_r(\phi + 2\pi) = P_r(\phi)$
(c) For each $r < 1$,

$$\int_{-\pi}^{\pi} P_r(t)\,dt = 2\pi.$$

Of these, (a) and (b) are evident, and (c) follows from the series development for $P_r(\phi)$.

If $F \in L_p[-\pi, \pi]$, it is convenient to suppose F *defined on all of* $\mathbb{R}$ *by periodicity*, $F(t + 2\pi) = F(t)$. *We henceforth assume this.* First we have converses to the representation theorems given in Section C.

Theorem *If* $p \geq 1$ *and* $F \in L_p[-\pi, \pi]$ *and*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)F(t)\,dt,$$

then $U(z)$ *is harmonic in* $|z| < 1$ *and*

$$\int_{-\pi}^{\pi} |U(re^{i\theta})|^p\,d\theta \leq \text{const.}, \qquad r < 1.$$

Proof Let

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}F(t)\,dt = A_n.$$

Then, for $0 \leq r < 1$,

$$U(re^{i\theta}) = \sum_{-\infty}^{\infty} A_n r^{|n|} e^{in\theta},$$

which is harmonic in $|z| < 1$ *by inspection*, because the series converges uniformly in the interior (meaning *uniformly on compact subsets* – complex variable language!) of that region. (If F is *real*, the series is clearly the real part of an analytic function which can be easily written down.)

Given $r < 1$, by property (b) and 2π-periodicity of F we can also write

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta - s)P_r(s)\,ds.$$

Now take $G \in L_q[-\pi, \pi]$, $\|G\|_q = 1$, so that (with any given fixed r – of course G will depend on r)

$$\left[\int_{-\pi}^{\pi} |U(re^{i\theta})|^p\,d\theta\right]^{1/p} = \int_{-\pi}^{\pi} U(re^{i\theta})G(\theta)\,d\theta.$$

By Fubini's theorem, the integral on the right is

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_r(s)F(\theta - s)G(\theta)\,d\theta\,ds$$

which is in modulus

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s)\,||F||_p\,||G||_q\,\mathrm{d}s = ||F||_p$$

by choice of G and property (c).

In fine,

$$\int_{-\pi}^{\pi} |U(re^{i\theta})|^p\,\mathrm{d}\theta \leq ||F||_p^p$$

and we are done.

Theorem *Let μ be a finite signed measure on $[-\pi, \pi]$. Then*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\,\mathrm{d}\mu(t)$$

is harmonic in $|z| < 1$ and

$$\int_{-\pi}^{\pi} |U(re^{i\theta})|\,\mathrm{d}\theta \leq \text{const.}, \quad r < 1.$$

Proof *Harmonicity* is established as above. Given $r < 1$, let $G \in L_\infty$, $||G||_\infty = 1$, be such that

$$\int_{-\pi}^{\pi} |U(re^{i\theta})|\,\mathrm{d}\theta = \int_{-\pi}^{\pi} G(\theta)U(re^{i\theta})\,\mathrm{d}\theta.$$

The right-hand integral is, by Fubini's theorem, equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_r(\theta - t)G(\theta)\,\mathrm{d}\theta\,\mathrm{d}\mu(t)$$

which, by properties (a), (b) and (c) is modulus

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\,||G||_\infty\,\mathrm{d}\theta\,|\mathrm{d}\mu(t)| = ||G||_\infty \int_{-\pi}^{\pi} |\mathrm{d}\mu(t)| = \int_{-\pi}^{\pi} |\mathrm{d}\mu(t)|.$$

We are done.

2. Elementary study of boundary behaviour

The Poisson kernel $P_r(\theta)$ has a *fourth property*:

(d) Given any $\delta > 0$,

$$P_r(\theta) \to 0 \quad \text{uniformly for} \quad \delta \leq |\theta| \leq \pi \quad \text{as } r \to 1.$$

This is obvious from the formula for $P_r(\theta)$.

Theorem *Let F be continuous on $\mathbb{R}$ and $F(t + 2\pi) = F(t)$. Let*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)F(t)\,\mathrm{d}t.$$

Then $U(z) \to F(\phi)$ as $z \to e^{i\phi}$, and the convergence is uniform in ϕ.

Proof The result goes back to Poisson himself, who *thought* it showed that the Fourier series of a function *converges* to that function (it doesn't show that!). Write

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\theta - t)P_r(t)\,dt.$$

Given any ϕ, we have, by property (c),

$$F(\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(\phi)P_r(t)\,dt.$$

Therefore

$$U(re^{i\theta}) - F(\phi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (F(\theta - t) - F(\phi))\,P_r(t)\,dt.$$

Let $\delta < \pi/2$ be such that $|F(s) - F(\phi)| < \epsilon$ for $|s - \phi| < 2\delta$; δ depends only on ϵ and not on ϕ here by (uniform!) continuity of F.

Write the last right hand integral as a sum of two:

$$|U(re^{i\theta}) - F(\phi)| \leq \frac{1}{2\pi}\int_{|t|\leq\delta} |F(\theta - t) - F(\phi)|P_r(t)\,dt$$
$$+ \frac{1}{2\pi}\int_{\delta\leqslant|t|\leqslant\pi} |F(\theta - t) - F(\phi)|P_r(t)\,dt.$$

If $|\theta - \phi| < \delta$, the first integral on the right is

$$\leq \frac{\epsilon}{2\pi}\int_{|t|\leq\delta} P_r(t)\,dt < \epsilon.$$

Let M be a bound on $|F(t)|$. Then the second integral is

$$\leq \frac{M}{\pi}\int_{\delta\leqslant|t|\leqslant\pi} P_r(t)\,dt$$

which is $< \epsilon$, say, if r is close enough to 1, by property (d).

So $|U(re^{i\theta}) - F(\phi)| < 2\epsilon$ if $|\theta - \phi| < \delta$ and r is close enough to 1. Q.E.D.

Remark Properties (a), (b), (c) and (d) together constitute the so-called *approximate identity property* of $(1/2\pi)P_r(\theta)$. The above theorem holds good *because of them* – all kinds of other kernels besides the Poisson kernel would work to yield similar theorems.

Theorem *Let $F \in L_1(-\pi, \pi)$, and suppose $F(t)$ is continuous at θ_0. Then*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)F(t)\,dt$$

tends to $F(\theta_0)$ as $re^{i\theta}$ tends to $e^{i\theta_0}$.

Proof Similar to that of the above theorem.

Theorem *Let $F \in L_p$, $1 \le p < \infty$ (sic!) and let*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)F(t)\,\mathrm{d}t.$$

Then

$$\int_{-\pi}^{\pi} |U(re^{i\theta}) - F(\theta)|^p\,\mathrm{d}\theta \to 0$$

as $r \to 1$; i.e., $U(re^{i\theta})$ approaches $F(\theta)$ in L_p norm as $r \to 1$.

Proof Write $F_r(\theta) = U(re^{i\theta})$; then

$$F_r(\theta) - F(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} [F(\theta - t) - F(\theta)]\,P_r(t)\,\mathrm{d}t.$$

Using properties (a) and (c) (we think of $F_r(\theta) - F(\theta)$ as a limit of convex combinations of the functions $F(\theta - t) - F(\theta)$, taking t as a parameter and θ as the variable), we have, by an evident generalization of the triangle inequality

$$\left(\int_{-\pi}^{\pi} |F_r(\theta) - F(\theta)|^p\,\mathrm{d}\theta\right)^{1/p} \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} |F(\theta - t) - F(\theta)|^p\,\mathrm{d}\theta\right)^{1/p} \cdot P_r(t)\,\mathrm{d}t.$$

That is, if we write

$$\Phi(t) = \left(\int_{-\pi}^{\pi} |F(\theta - t) - F(\theta)|^p\,\mathrm{d}\theta\right)^{1/p},$$

$$\|F_r - F\|_p \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(t)P_r(t)\,\mathrm{d}t.$$

But $\Phi(t) \to 0$ as $t \to 0$! *That is because translation is continuous in the L_p norm for* $1 \le p < \infty$. This, in turn, follows from the rudiments of real variable theory as follows: given $F \in L_p(-\pi, \pi)$ and $\epsilon > 0$ take a continuous G, periodic of period 2π, with $\|F - G\|_p < \epsilon$. Then

$$\int_{-\pi}^{\pi} |G(\theta - t) - G(\theta)|^p\,\mathrm{d}\theta$$

is obviously $< \epsilon^p$ for $|t| < \delta$, say, by uniform continuity, so

$$\|F(\theta - t) - F(\theta)\|_p < 3\epsilon \quad \text{for} \quad |t| < \delta.$$

In particular, $\Phi(t)$ is *continuous at* 0 *where it equals zero.*

So, by a previous result, $\|F_r - F\|_p \to 0$ as $r \to 1$.

If $p = \infty$ all we have is weak* convergence:

Theorem *If $F \in L_\infty$, and*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)F(t)\,\mathrm{d}t,$$

then $U(re^{i\theta}) \to F(\theta)$ w* *as* $r \to 1$.

Proof Take any $G \in L_1(-\pi, \pi)$. We are to prove that

$$\int_{-\pi}^{\pi} U(re^{i\theta})G(\theta)\,\mathrm{d}\theta \to \int_{-\pi}^{\pi} F(\theta)G(\theta)\,\mathrm{d}\theta$$

for $r \to 1$. But this is true, because ($P_r(\phi)$ being even!)

$$\int_{-\pi}^{\pi} G(\theta)P_r(\theta - t)\,\mathrm{d}\theta = \int_{-\pi}^{\pi} P_r(t-\theta)G(\theta)\,\mathrm{d}\theta$$

tends in L_1 norm to $G(t)$ as $r \to 1$ by the preceding theorem. We need then only apply Fubini's theorem.

Similarly

Theorem *Let*

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\,\mathrm{d}\mu(t)$$

with μ *a finite signed measure on* $[-\pi, \pi]$. *Then* $U(re^{i\theta})\,\mathrm{d}\theta \to \mathrm{d}\mu(\theta)$ w* *as* $r \to 1$, *i.e., for any continuous* $G(\theta)$, *periodic and of period* 2π,

$$\int_{-\pi}^{\pi} U(re^{i\theta})G(\theta)\,\mathrm{d}\theta \to \int_{-\pi}^{\pi} G(\theta)\,\mathrm{d}\mu(\theta)$$

as $r \to 1$.

Proof Use Fubini's theorem together with the first result of this subsection.

3. Deeper study of boundary behaviour; Fatou's theorem

If $U(z)$, harmonic in $|z| < 1$, has one of the representations

$$U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)F(t)\,\mathrm{d}t, \quad F \in L_p, \qquad U(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\,\mathrm{d}\mu(t),$$

we have still to discuss the pointwise behaviour of $U(z)$ as z tends to points $e^{i\theta}$ on the boundary of the unit circle. Study of such behaviour cannot proceed on the basis of the approximate identity properties (a)-(d) alone, but requires a more detailed examination of $P_r(\theta)$.

Both representations written above for $U(re^{i\theta})$ are subsumed in the second one, for if $F \in L_p(-\pi, \pi)$, and we take $\mathrm{d}\mu(\theta) = F(\theta)\,\mathrm{d}\theta$, then μ *is* in fact a finite signed measure on $[-\pi, \pi]$. In dealing with such a measure, it is convenient to introduce the function $\mu(\theta)$ of bounded variation on $[-\pi, \pi]$ given by

$$\mu(\theta) = \int_{0}^{\theta} \mathrm{d}\mu(t)$$

(with usual interpretation of the integral if $\theta < 0$). Then we have the