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# Independent Random Variables and Rearrangement Invariant Spaces

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# PREFACE

Rearrangement invariant (r.i.) spaces were first considered in connection with questions arising in the theory of interpolation of linear operators. Now they are the object of intensive study. The main property of r.i. spaces is that the norm of any element $X$ depends only on the distribution of $X$. This is why probabilistic methods have been used successfully.

The purpose of this monograph is to study sequences of independent random variables (r.v.) as sequences of elements of r.i. spaces. Similar problems have been discussed by many authors (see **[15]**, **[17]**, **[23]**, **[24]**, **[33]** et al.). The work consists of four chapters. The first chapter contains some well known notation and results from probability theory and the theory of r.i. spaces which are used in the following chapters. Besides, we introduce a new term, which we will call the Kruglov property.

Let $F$ be a probability distribution on $\mathbf{R}$ and $\Pi(F)$ be the corresponding compound Poisson distribution, i.e. the distribution with the characteristic function

$$g(t) = \exp\left(\int_{-\infty}^{\infty} (e^{itx} - 1)dF(x)\right).$$

Kruglov **[30]** described the class of positive functions $\Phi(x)$ on $\mathbf{R}$ for which the conditions

$$\int_{-\infty}^{\infty} \Phi(x)dF(x) < \infty, \quad \int_{-\infty}^{\infty} \Phi(x)d(\Pi(F))(x) < \infty$$

are equivalent. We say that a r.i. space $\mathbf{E}$ has the Kruglov property ($\mathbf{E} \in \mathbf{K}$) if for any r.v.s $X$ and $Y$ with the distributions $F$ and $\Pi(F)$ respectively the conditions $X \in \mathbf{E}$ and $Y \in \mathbf{E}$ are equivalent.

For instance, $L_p \in \mathbf{K}$ if $p < \infty$ . The Kruglov property plays an important role in our considerations because it permits us to compare sequences of independent and disjoint r.v.s.

In Chapter 2 we prove several results about analogs of the Rosenthal and the von Bahr and Esseen inequalities (see **[2]** and **[50]**). One of them is the following: if $\mathbf{E} \in \mathbf{K}$ and some analog of Rosenthal's inequality holds, then $\mathbf{E} = L_p$ for some $p \in (2, \infty)$.

Another result states that under the assumption $\mathbf{E} \in \mathbf{K}$ an analog of the von Bahr and Esseen estimate is fulfilled in $\mathbf{E}$ if and only if a similar inequality is true for mutually disjoint r.v.s, i.e. $\mathbf{E}$ satisfies the upper $p$-estimate. We note that similar results have been proved in **[14]** and **[24]**.

In Chapter 3 we study sequences of independent identically distributed r.v.s. The results on the conditions under which such a sequence generates the subspace in $\mathbf{E}$ isomorphic to the space $l_p$ are obtained.

In Chapter 4 the question of the complementability in **E** of the subspace generated by a sequence of independent r.v.s is considered. Similar problems have been studied by many authors (see [**17**], [**33**], [**49**] et al.).
Each chapter has its own enumeration. With references to a statement or formula of another chapter the number of the latter is assigned, but it is not added if the reference is to the same chapter.