CHAPTER I

PRELIMINARIES

1. Rearrangement invariant spaces

Let $(\Omega, \mathcal{F}, P)$ be a probability space, i.e. a measure space such that $P(\Omega) = 1$. A measurable function X on Ω is called a *random variable* (*r.v.*). We write $X \stackrel{d}{=} Y$ if these r.v.s are identically distributed. This means that $P\{X < x\} = P\{Y < x\}$ for every $x \in \mathbf{R}$. The *mean value* of a r.v. X is defined by the formula

$$EX = \int_\Omega X(\omega) dP(\omega).$$

The *distribution* of a r.v. X is the measure F on $\mathbf{R}$ defined by the formula $F(h) = P\{X \in h\}$ for every measurable set $h \subset \mathbf{R}$. We write $X \in \mathcal{L}(F)$ if X has the distribution F. The function

$$f(t) = \int_{-\infty}^{\infty} \exp(itx) dF(x)$$

is called the *characteristic function* of the r.v. X.

Definition 1. *A Banach space* $\mathbf{E}$ *of random variables defined on* $(\Omega, \mathcal{F}, P)$ *is said to be rearrangement invariant (r.i.) if the following conditions hold:*
(i) if $|X| \leq |Y|$ *and* $Y \in \mathbf{E}$, *then* $X \in \mathbf{E}$ *and* $\|X\|_\mathbf{E} \leq \|Y\|_\mathbf{E}$;
(ii) if $X \stackrel{d}{=} Y$ *and* $Y \in \mathbf{E}$, *then* $X \in \mathbf{E}$ *and* $\|X\|_\mathbf{E} = \|Y\|_\mathbf{E}$.

The main information on r.i. spaces is contained in the book [**29**]. The spaces $L_p(\Omega)$ $(1 \leq p \leq \infty)$ are rearrangement invariant. The Lorentz spaces $L_{p,q}(\Omega)$ form a wider class of r.i. spaces which consist of all r.v.s X such that

$$\|X\|^*_{p,q} = \left(\int_0^\infty (P\{|X| \geq x\})^{q/p} dx^q \right)^{1/q} < \infty \tag{1}$$

if $1 \leq p, q < \infty$, and,

$$\|X\|^*_{p,\infty} = \sup \left\{ x (P\{|X| \geq x\})^{1/p} : x > 0 \right\} < \infty . \tag{2}$$

The functionals (1) and (2) are not norms in general, but they are equivalent to some norms (see [**52**]). We have $L_{p,p} = L_p$.

Orlicz spaces form another class of r.i. spaces. 1Let $N(x)$ be a convex even function on $\mathbf{R}, N(0) = 0$. The Orlicz space $L_N(\Omega)$ consists of all r.v.s X such that $EN(\lambda^{-1}X) < \infty$ for some $\lambda > 0$. The norm is defined by the formula (see [**28**])

$$||X||_{L_N} = \inf\{\lambda > 0 : EN(\lambda^{-1}X) < 1\}. \quad (3)$$

In the sequel the probability space $(\Omega, \mathcal{F}, P)$ is assumed to be non-atomic. The *indicator* of a set h is denoted by I_h.

The following statements are proved in [**29**].

Proposition 1. *For every r.i. space* $\mathbf{E}$

$$L_\infty(\Omega) \subset \mathbf{E} \subset L_1(\Omega)$$

and for each r.v. $X \in L_\infty(\Omega)$

$$||X||_{L_1(\Omega)} \le \frac{||X||_{\mathbf{E}}}{||I_\Omega||_{\mathbf{E}}} \le ||X||_{L_\infty(\Omega)}.$$

Proposition 2. *Let* $Y \in \mathbf{E}$ *and* $P\{|X| \ge x\} \le CP\{|Y| \ge x\}$ *for all* $x > 0$, *where* C *is a constant. Then* $X \in \mathbf{E}$ *and* $||X||_{\mathbf{E}} \le \max\{1, C\}||Y||_{\mathbf{E}}$.

The *dual* or *associated* space $\mathbf{E}'$ of the r.i. space $\mathbf{E}$ is the set of all r.v.s Y such that

$$||Y||_{\mathbf{E}'} \stackrel{def}{=} \sup\{EXY : X \in \mathbf{E}, ||X||_{\mathbf{E}} \le 1\} < \infty.$$

It is well known (see [**29**]) that $\mathbf{E}'$ is a r.i. space. Write $\mathbf{E}'' = (\mathbf{E}')'$. We have $\mathbf{E} \subset \mathbf{E}''$ and $||X||_{\mathbf{E}''} \le ||X||_{\mathbf{E}}$ for every $X \in \mathbf{E}$. If $X \in L_\infty(\Omega)$, then $||X||_{\mathbf{E}''} = ||X||_{\mathbf{E}}$. A r.i. space $\mathbf{E}$ is said to be *maximal*, if $\mathbf{E}'' = \mathbf{E}$.

The *decreasing rearrangement* of a r.v. X is the function on $(0, 1)$ defined by the formula

$$X^*(t) = \inf\{s > 0 : P\{|X| \ge s\} < t\}.$$

We write $X \prec Y$ for $X, Y \in L_1(\Omega)$ if for each $0 \le t \le 1$

$$\int_0^t X^*(s)ds \le \int_0^t Y^*(s)ds.$$

The following assertions are proved in [**29**].

Proposition 3. *Let a r.i. space* $\mathbf{E}$ *be maximal or separable,* $Y \in \mathbf{E}$ *and* $X \prec Y$. *Then* $X \in \mathbf{E}$ *and* $||X||_{\mathbf{E}} \le ||Y||_{\mathbf{E}}$.

Proposition 4 (Calderon–Mityagin's Theorem). *Suppose that a r.i. space* $\mathbf{E}$ *has the following property: if* $Y \in \mathbf{E}$ *and* $X \prec Y$ *then* $X \in \mathbf{E}$ *and*

$||X||_{\mathbf{E}} \leq ||Y||_{\mathbf{E}}$. *Let a linear operator T be bounded in $L_1(\Omega)$ and $L_\infty(\Omega)$. Then T is bounded in $\mathbf{E}$ and*

$$||T||_{\mathbf{E}\to\mathbf{E}} \leq \max\left\{||T||_{L_1(\Omega)\to L_1(\Omega)},\ ||T||_{L_\infty(\Omega)\to L_\infty(\Omega)}\right\}.$$

Let X_n and X be r.v.s with the distributions F_n and F respectively. The sequence X_n is said to be *weakly convergent to* X if for every continuous bounded function $g(x)$ on $\mathbf{R}$

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} g(x)dF_n(x) = \int_{-\infty}^{\infty} g(x)dF(x).$$

Weak convergence is equivalent to each of the following conditions:
(i) $F_n(x) \to F(x)$ for all $x \in \mathbf{R}$ on which F is continious ;
(ii) $f_n(t) \to f(t)$ for each $t \in \mathbf{R}$, where $f_n(t)$ and $f(t)$ are the corresponding characteristic functions (see [**51**]).

Proposition 5. *Let $\mathbf{E}$ be a maximal r.i. space and $X_n \in \mathbf{E}$. Suppose X_n is weakly convergent to X and*

$$\sup_n ||X_n||_{\mathbf{E}} = C < \infty.$$

Then $X \in \mathbf{E}$ and $||X||_{\mathbf{E}} \leq C$.

Proof: Let $Y \in \mathbf{E}'$ and $||Y||_{\mathbf{E}'} \leq 1$. According to Proposition 1, $E|Y| < \infty$. From here $Y^* \in L_1(0,1)$ and the function

$$\Psi_Y(t) = \int_0^t Y^*(s)ds$$

is well defined on $(0,1)$. It is clear that

$$\int_0^1 X^*(t)Y^*(t)dt = \int_0^1 X^*(t)d\Psi_Y(t).$$

We use the formulae (see [**29**], Ch. 2)

$$\int_0^1 X^*(t)d\Psi_Y(t) = \int_0^\infty \Psi_Y(P\{|X| \geq x\})dx$$

and

$$||X||_{\mathbf{E}''} = \sup\left\{\int_0^1 X^*(t)Y^*(t)dt : ||Y||_{\mathbf{E}'} \leq 1\right\}.$$

Fatou's lemma and these equalities imply that

$$\int_0^\infty \Psi_Y(P\{|X| \geq x\})dx \leq \sup_n \int_0^\infty \Psi_Y(P\{|X_n| \geq x\})dx$$
$$= \sup_n \int_0^1 X_n^*(t)Y^*(t)dt \leq \sup_n ||X_n||_{\mathbf{E}} = C.$$

So,

$$\sup\left\{\int_0^1 X^*(t)Y^*(t)dt : \ ||Y||_{\mathbf{E}'} \leq 1\right\} \leq C < \infty.$$

Hence $X \in \mathbf{E}'' = \mathbf{E}$ and $||X||_{\mathbf{E}} \leq C$. □

If $\mathbf{E}$ is a r.i. space, then the norm $||I_h||_{\mathbf{E}}$ depends on $P(h)$ only. Therefore the function

$$\phi_{\mathbf{E}}(t) = ||I_h||_{\mathbf{E}} \quad (P(h) = t)$$

is well defined on $[0, 1]$. It is called *the fundamental function* of $\mathbf{E}$. The next statement follows from the results of [**29**], Ch. 2.

Proposition 6. *If*

$$b_{\mathbf{E}}(X) \overset{def}{=} \int_0^\infty \phi_{\mathbf{E}}(P\{|X| \geq x\})dx < \infty,$$

then $X \in \mathbf{E}$ and $||X||_{\mathbf{E}} \leq a(\mathbf{E})b_{\mathbf{E}}(X)$, where $a(\mathbf{E})$ depends on $\mathbf{E}$ only.

2. The function of dilatation and Boyd indices

For the r.v. X and $t > 0$ let $B_t(X)$ denote the set of all r.v.s Y such that $P\{x \leq Y < y\} \leq tP\{x \leq X < y\}$ for every $x < y, xy > 0$. If $Y \in B_t(X)$, then

$$P\{|Y| \geq x\} \leq tP\{|X| \geq x\}$$

for all positive x. Proposition 2 implies that if $X \in \mathbf{E}$, then $B_t(X) \subset \mathbf{E}$ for every $t > 0$. Put

$$\gamma_{\mathbf{E}}(t) = \sup\left\{\frac{||X||_{\mathbf{E}}}{||Y||_{\mathbf{E}}} : Y \in B_t(X),\ X \in \mathbf{E},\ X \neq 0\right\}.$$

We call $\gamma_{\mathbf{E}}(t)$ the *function of dilatation* of the r.i. space $\mathbf{E}$. It is obvious that $\gamma_{\mathbf{E}}(t)$ is non-decreasing.

Suppose that the probability space is $[0, 1]$ with Lebesgue measure and for a r.v. $X(s)$ on $[0, 1]$ putp

$$D_tX(s) = \begin{cases} X(s/t) & \text{if } s < t, \\ 0 & \text{if } t < s \leq 1. \end{cases}$$

The operator D_t is bounded in every r.i. space $\mathbf{E}$ (see [**29**]). It is easy to verify that $\gamma_{\mathbf{E}}(t) = ||D_t||_{\mathbf{E}\to\mathbf{E}}$.

Proposition 7. *The following estimates are true:*

$$\min\{1,t\} \le \gamma_{\mathbf{E}}(t) \le \max\{1,t\}.$$

These inequalities follow directly from the definition and the obvious relations $\gamma_{\mathbf{E}}(t) \le 1 = \gamma_{\mathbf{E}}(1)$ if $t \le 1$ and $\gamma_{\mathbf{E}}(t) \ge 1$ if $t > 1$.

The *Boyd indices* are defined by the formulae

$$\alpha(\mathbf{E}) = \lim_{t\to 0} \frac{\log(\gamma_{\mathbf{E}}(t))}{\log(t)}, \quad \beta(\mathbf{E}) = \lim_{t\to\infty} \frac{\log(\gamma_{\mathbf{E}}(t))}{\log(t)}.$$

These limits exist and $0 \le \alpha(\mathbf{E}) \le \beta(E) \le 1$ (see [**29**], Ch.2). We have $\alpha(L_{p,q}) = \beta(L_{p,q}) = 1/p$.

Proposition 8. *For each $\epsilon > 0$ there exist positive constants a and b such that $\gamma_{\mathbf{E}}(t) \le t^{\alpha(\mathbf{E})-\epsilon}$ if $t < a$ and $\gamma_{\mathbf{E}}(t) \le t^{\beta(\mathbf{E})+\epsilon}$ if $t > b$.*

Proof: See [**29**], Ch. 2. □

The following assertion is well known (see [**23**]).

Proposition 9. *If $\alpha(\mathbf{E}) > 0$, then $\mathbf{E} \supset L_p(\Omega)$ for all $p > 1/\alpha(\mathbf{E})$. If $\beta(\mathbf{E}) < 1$, then $\mathbf{E} \subset L_q(\Omega)$ for each $q < 1/\beta(\mathbf{E})$.*

3. Independent random variables

Definition 2. *Let H be a set of indices. The r.v.s $\{X_h\}$, $h \in H$, are said to be independent, if for every finite set $h_1, \dots, h_m$ and each $a_k < b_k$ $(1 \le k \le m)$*

$$P\{a_k \le X_{h_k} \le b_k, 1 \le k \le m\} = \prod_{k=1}^{m} P\{a_k \le X_{h_k} \le b_k\}.$$

Proposition 10. *Let r.v.s X and Y be independent, $\mathbf{E}$ be a r.i. space and $X + Y \in \mathbf{E}$. Then $X \in \mathbf{E}$ and $Y \in \mathbf{E}$.*

Proof: Let α be a *median* of X. It means $P\{X \le \alpha\} \ge 1/2$ and $P\{X \ge \alpha\} \ge 1/2$ (see [**35**]). Since X and Y are independent, then for every $x > 0$

$$\begin{aligned} &P\{|X+Y| \ge x\} \\ &\ge P\{X \ge x-\alpha, Y \ge \alpha\} + P\{X \le -x-\alpha, Y \le \alpha\} \\ &\ge \frac{1}{2}P\{X \ge x-\alpha\} + \frac{1}{2}P\{X \le -x-\alpha\} \\ &\ge \frac{1}{2}P\{|X| \ge x + |\alpha|\}. \end{aligned}$$

Using Proposition 2, we get $X \in \mathbf{E}$. □

Proposition 11. *For each r.i. space* $\mathbf{E}$ *there exists a constant* $C(\mathbf{E}) > 0$ *such that for any independent r.v.s* $X, Y \in \mathbf{E}$, $EY = 0$

$$||X + Y||_{\mathbf{E}} \geq C(\mathbf{E})||X||_{\mathbf{E}}.$$

Proof: Suppose that this assertion is not true for some r.i. space $\mathbf{E}$. Then there exist r.v.s $X_n, Y_n \in \mathbf{E}$ with the following properties:

1) the r.v.s $\{X_k\}_{k=1}^{\infty} \bigcup \{Y_k\}_{k=1}^{\infty}$ are independent;
2) $EY_k = 0$;
3) $||X_k||_{\mathbf{E}} = 1$;
4) $||X_k + Y_k||_{\mathbf{E}} \leq 2^{-k}$.

According to 4), the series $\sum_{k=1}^{\infty}(X_k + Y_k)$ is absolutely convergent in $\mathbf{E}$ and, therefore, in $L_1(\Omega)$. Let β be the σ-algebra generated by the sequence $\{X_k\}_{k=1}^{\infty}$. Since the operator E^{β} of conditional expectation is bounded in $L_1(\Omega)$, the conditions 1) and 2) imply for each $m < n$ (see [**35**])

$$E^{\beta}\left(\sum_{k=m}^{n}(X_k + Y_k)\right) = \sum_{k=m}^{n} X_k + \sum_{k=m}^{n} EY_k = \sum_{k=m}^{n} X_k.$$

Hence the series $\sum_{k=1}^{\infty} X_k$ and $\sum_{k=1}^{\infty} Y_k = \sum_{k=1}^{\infty}(X_k + Y_k) - \sum_{k=1}^{\infty} X_k$ are convergent in $L_1(\Omega)$. According to 1) and 2), the second series is convergent almost surely (see [**35**]). Hence $Y_k \to 0$ almost surely.

Let $h_k(\epsilon) = \{|Y_k| < \epsilon\}$, where $0 < \epsilon < 1$. We have $P(h_k(\epsilon)) \geq 1 - \epsilon$ for sufficiently large k and

$$|X_k + Y_k| I_{h_k(\epsilon)} \geq ||X_k| - \epsilon| I_{h_k(\epsilon)}.$$

By virtue of independence, for each $x > 0$

$$P\left\{||X_k| - \epsilon| I_{h_k(\epsilon)} \geq x\right\} = P(h_k(\epsilon)) P\left\{||X_k| - \epsilon| \geq x\right\}.$$

From here and Proposition 2

$$||(|X_k| - \epsilon) I_{h_k(\epsilon)}||_{\mathbf{E}} \geq P(h_k(\epsilon))|||X_k| - \epsilon||_{\mathbf{E}}.$$

Hence for sufficiently large k

$$\begin{aligned} ||X_k + Y_k||_{\mathbf{E}} &\geq ||(X_k + Y_k) I_{h_k(\epsilon)}||_{\mathbf{E}} \geq ||(|X_k| - \epsilon) I_{h_k(\epsilon)}||_{\mathbf{E}} \\ &\geq P(h_k(\epsilon))|||X_k| - \epsilon||_{\mathbf{E}} \geq (1 - \epsilon)(||X_k||_{\mathbf{E}} - \epsilon) \\ &= (1 - \epsilon)^2. \end{aligned}$$

This contradicts 4). □

Remark. If $\mathbf{E}$ is separable or maximal, then $C(\mathbf{E}) = 1$. Indeed, let β be the σ-algebra generated by X. Proposition 4 implies that the operator E^β is bounded in $\mathbf{E}$ and has norm equal to 1. Hence

$$\|X+Y\|_{\mathbf{E}} \geq \|E^\beta(X+Y)\|_{\mathbf{E}} = \|X+EY\|_{\mathbf{E}} = \|X\|_{\mathbf{E}}.$$

For the spaces $L_p(\Omega)$ $(1 \leq p \leq \infty)$ this statement is well known (see [**35**]).

4. Probability inequalities

Here we present some well known estimates which will be used in the following chapters.

1. Paley and Zygmund's inequality [26]. *Let $\{X_k\}_{k=1}^\infty$ be independent r.v.s such that $EX_k^4 < \infty$ and $EX_k = 0$. Suppose*

$$D = \sup\left\{\frac{EX_k^4}{(EX_k^2)^2} < \infty\right\}.$$

Then for every $a_k \in \mathbf{R}$ and $n \in \mathbf{N}$

$$P\left\{\left|\sum_{k=1}^n a_k X_k\right| \geq \lambda \left(\sum_{k=1}^n a_k^2\right)^{1/2}\right\} \geq \eta,$$

where $\eta = (1-\lambda^2)\min\{1/3, 1/D\}$.

2. Bernstein's inequality [43]. *Let $\{X_k\}_{k=1}^\infty$ be independent r.v.s, $|X_k| \leq C$ and $EX_k = 0$. Then*

$$P\left\{\left|\sum_{k=1}^n a_k X_k\right| \geq 2Cx \left(\sum_{k=1}^n a_k^2\right)^{1/2}\right\} \leq \exp(-x^2)$$

for all $x > 0$, $a_k \in \mathbf{R}$ and $n \in \mathbf{N}$.

3. Prokhorov's 'arcsinh' inequality [46]. *Let $\{X_k\}_{k=1}^\infty$ be independent r.v.s, $|X_k| \leq C$ and $EX_k = 0$. Let $\sigma^2 = \sum_{k=1}^n EX_k^2$. Then for every $x > 0$*

$$P\left\{\left|\sum_{k=1}^n X_k\right| \geq x\right\} \leq 2\exp\left(-\frac{x}{2C}\operatorname{arcsinh}\frac{Cx}{\sigma^2}\right).$$

A r.v. X is said to be *symmetric* if $P\{X \geq x\} = P\{X \leq -x\}$ for all $x > 0$. If F is a probability distribution on $\mathbf{R}$, then $\Pi(F)$ denotes the corresponding Poisson distribution, i.e. the distribution with the characteristic function

$$g(t) = \exp\left(\int_{-\infty}^{\infty}(e^{itx}-1)dF(x)\right). \tag{4}$$

Let us recall we write $X \in \mathcal{L}(F)$ if F is the distribution of the r.v. X.

4. Prokhorov's inequality [45]. *Let $\{X_k\}_{k=1}^n$ be independent symmetric r.v.s with distributions F_k and let the r.v.s $Y_k \in \mathcal{L}(\Pi(F_k))$ be independent. Then for every $x > 0$*

$$P\left\{\left|\sum_{k=1}^n X_k\right| \ge x\right\} \le 8P\left\{\left|\sum_{k=1}^n Y_k\right| \ge \frac{x}{2}\right\}.$$

5. Kwapien and Richlik's inequality [53]. *Let $\{X_k\}_{k=1}^\infty$ and $\{Y_k\}_{k=1}^\infty$ be sequences of independent symmetric r.v.s such that $P\{|X_k| \ge x\} \le AP\{B|Y_k| \ge x\}$ for all $x > 0$ and $k \in \mathbf{N}$. Then for every $a_k \in \mathbf{R}$, $x > 0$ and $n \in \mathbf{N}$*

$$P\left\{\left|\sum_{k=1}^n a_k X_k\right| \ge x\right\} \le 2AP\left\{AB\left|\sum_{k=1}^n a_k Y_k\right| \ge x\right\}.$$

5. Disjoint random variables

We say the r.v.s X and Y are *disjoint*, if $XY = 0$.

Definition 3. *[33]. A r.i. space $\mathbf{E}$ is said satisfy the upper, respectively the lower, r-estimate if there exists a positive constant A such that for each collection of mutually disjoint r.v.s $\{X_k\}_{k=1}^n \subset \mathbf{E}$*

$$\left\|\sum_{k=1}^n X_k\right\|_{\mathbf{E}} \le A\left(\sum_{k=1}^n \|X_k\|_{\mathbf{E}}^r\right)^{1/r},$$

respectively

$$\left\|\sum_{k=1}^n X_k\right\|_{\mathbf{E}} \ge A\left(\sum_{k=1}^n \|X_k\|_{\mathbf{E}}^r\right)^{1/r}.$$

The following assertions are known. For completeness the proofs are given.

Proposition 12. *Let $1 \le p < \infty$, $1 \le q \le \infty$ and $r = \min\{p, q\}$. The Lorentz space $L_{p,q}(\Omega)$ satisfies the upper r-estimate and does not satisfy the upper s-estimate for $s > r$.*

Proof: Let $\{X_k\}_{k=1}^n \subset \mathbf{E}$ be mutually disjoint r.v.s. Then for every $x > 0$

$$P\left\{\left|\sum_{k=1}^n X_k\right| \ge x\right\} = \sum_{k=1}^n P\{|X_k| \ge x\}.$$

Suppose $q = \infty$. According to (2)

$$\left(\left\|\sum_{k=1}^{n} X_k\right\|_{p,\infty}^{*}\right)^{p} = \sup\left\{x^p P\left\{\left|\sum_{k=1}^{n} X_k\right| \geq x\right\} : x > 0\right\}$$
$$= \sup\left\{x^p \sum_{k=1}^{n} P\{|X_k| \geq x\} : x > 0\right\}$$
$$\leq \sum_{k=1}^{n} \left(\|X_k\|_{p,\infty}^{*}\right)^{p}.$$

Let $r = p \leq q < \infty$. From the formula (1)

$$\left(\left\|\sum_{k=1}^{n} X_k\right\|_{p,q}^{*}\right)^{p} = \left(\int_0^{\infty} \left(\sum_{k=1}^{n} P\{|X_k| \geq x\}\right)^{q/p} dx^q\right)^{p/q}.$$

Applying Minkowski's inequality with the exponent $\alpha = q/p$, we get the needed estimate.

Let's turn to the case $r = q < p$. We have

$$\left(\left\|\sum_{k=1}^{n} X_k\right\|_{p,q}^{*}\right)^{q} = \int_0^{\infty} \left(\sum_{k=1}^{n} P\{|X_k| \geq x\}\right)^{q/p} dx^q.$$

Since $|a+b|^t \leq |a|^t + |b|^t$ for $0 < t < 1$, then

$$\left(\sum_{k=1}^{n} P\{|X_k| \geq x\}\right)^{q/p} \leq \sum_{k=1}^{n} \left(P\{|X_k| \geq x\}\right)^{q/p}.$$

From here the desired estimate follows.

Let $s > r$. If a r.i. space $\mathbf{E}$ satisfies the upper s-estimate, then $\beta(\mathbf{E}) < 1/s$ (see [**33**]). Since $\beta(L_{p,q}) = 1/p$, then, if $r = p$, the space $L_{p,q}(\Omega)$ does not satisfy the upper s-estimate.

Suppose $r = q$. Then, as shown in [**40**], there exist mutually disjoint normed r.v.s $\{X_k\}_{k=1}^{\infty} \subset L_{p,q}(\Omega)$ such that for all $n \in \mathbf{N}$

$$\left\|\sum_{k=1}^{n} X_k\right\|_{p,q}^{*} \geq n,$$

where $C > 0$ is a constant. So, $L_{p,q}(\Omega)$ does not satisfy the upper s-estimate for $s > r = q$ and the proposition is proved. □

Now we consider Orlicz spaces. Let $N_i(x)$ $(i = 1, 2)$ be even convex functions on $\mathbf{R}$. We say that these functions are *equivalent* if there exist positive constants A, B and C such that $N_1(Ax) \leq N_2(x) \leq N_1(Bx)$ for $x > C$. It is shown in [**28**] that $L_{N_1}(\Omega) = L_{N_2}(\Omega)$ if and only if the functions $N_1(x)$ and $N_2(x)$ are equivalent.

Proposition 13. *Let a convex even function $N(x)$, $N(0) = 0$, be equivalent to $U(x) = |x|^r\Psi(x)$, where $r > 1$ and $\Psi(x)$ is convex and increasing on $(0,\infty)$. Then the space $L_N(\Omega)$ satisfies the upper r-estimate.*

Proof.: Without loss of generality $N(x) = |x|^r\Psi(x)$. Suppose $\{X_k\}_{k=1}^n \subset L_N(\Omega)$ are mutually disjoint random variables and $EN(\lambda_k^{-1}X_k) \le 1$ for some real λ_k $(1 \le k \le n)$. Put $t_k = \lambda_k(\sum_{j=1}^n \lambda_j^r)^{-1/r}$. Then

$$EN\left(\left(\sum_{j=1}^n \lambda_j^r\right)^{-1/r}\sum_{k=1}^n X_k\right) = \sum_{k=1}^n EN(t_k\lambda_k^{-1}X_k).$$

Since the function $\Psi(x)$ is even and increasing on $(0,\infty)$ we have $\Psi(tx) \le \Psi(x)$ for $0 < t < 1$ and every $x \in \mathbf{R}$. Therefore $EN(tX) = E(|tX|^r\Psi(tX)) \le |t|^r E(|X|^r\Psi(X)) = |t|^r EN(X)$. From here

$$EN\left(\left(\sum_{j=1}^n \lambda_j^r\right)^{-1/r}\sum_{k=1}^n X_k\right) \le \sum_{k=1}^n t_k^r EN(\lambda_k^{-1}X_k) \le \sum_{k=1}^n t_k^r = 1.$$

Using (3), we get

$$\left\|\sum_{k=1}^n X_k\right\|_{L_N} \le \left(\sum_{k=1}^n t_k^r\right)^{1/r}.$$

Applying (3) once more, we obtain the desired bound. □

6. The Kruglov property

In this section we investigate relations between independent r.v.s and mutually disjoint ones. A similar problem has been studied by Carothers and Dilworth [**14**], [**15**] and Johnson and Schechtman [**24**]. Our approach is different from theirs. It is based on the so-called Kruglov property of a r.i. space.

Kruglov proved the following statement [**30**].

Kruglov's Theorem. *Let $\Phi(x)$ be a non-negative and continuous function on $\mathbf{R}$ satisfying one of the following conditions:*

$$\Phi(x+y) \le B\Phi(x)\Phi(y), \quad \Phi(x+y) \le B(\Phi(x)+\Phi(y)),$$

where B is a constant. Suppose $X \in \mathcal{L}(F)$ and $Y \in \mathcal{L}(\Pi(F))$. Then the conditions $E\Phi(X) < \infty$ and $E\Phi(Y) < \infty$ are equivalent.

The question arises about the conditions under which the result of this type is true for r.i. spaces.