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Combinatorial Methods in Discrete Mathematics

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Preface

This book is addressed to those who are interested in combinatorial methods of discrete mathematics and their applications. A major part of the book can be used as a textbook on combinatorial analysis for students specializing in mathematics. The remaining part is suitable for use in special lectures and seminars for the advanced study of combinatorics. Those parts which are not intended for teaching include Sections 2.3, 3.6, 3.7, 5.3, 5.6, 5.8, 6.3, 6.4 and Subsection 5.1.3 where the material contains either special questions concerning applications of combinatorial methods or rather cumbersome derivations of asymptotic formulae. Of course, a course of studies in discrete mathematics can be biased towards asymptotic methods, where the selection of material can be different and where the above-mentioned sections become basic.

Some knowledge of algebra and set theory, summarized in Section 1.1, is assumed. To understand the derivations of asymptotic formulae, the reader must be familiar with those results of complex analysis usually included in standard courses for students specializing in mathematics.

For the convenience of those readers who are interested in the separate questions contained in the book, I have attempted to make the presentation of each chapter self-contained and, for the most part, independent of the other chapters.

As is usual, I acknowledge those authors whose results are presented in the book and provide the corresponding citation. The list of references is given at the end of the book.

The method of citation is unified. Citations of theorems, lemmas, corollaries, formulae, etc., include the chapter number, section number and own number within the section. For example, Theorem 1.2.3 is theorem number 3 in Section 2 of Chapter 1.

V. N. Sachkov

Preface to the English edition

A few additions, which did not change the structure and level of presentation, have been included in the preparation of this English translation. Chapter 1 is supplemented by Section 1.8. In Chapter 2 insertions have been added to Subsection 2.1.3; this chapter is supplemented by Subsection 2.2.4 and Section 2.7; Section 2.6 is supplemented by Subsections 2.6.2 and 2.6.6. In Chapter 3, Sections 3.1, 3.3 and 3.6 are supplemented, and a new section, Section 3.4, is introduced. In Chapter 5, Subsection 5.2.6, item (g) in Section 5.4 and Section 5.9 have been added. Subsection 6.1.3 has been added to Chapter 6. In the main, the option of using certain parts of the book as a textbook, as well as the independence of the chapters, is preserved.

After the Russian edition of the book had appeared, a number of significant monographs on combinatorics and closely related problems were published. These books and some papers have been included in the Bibliography.

I am grateful to Professor B. Bollobás for his kind suggestion to Cambridge University Press to publish this book in English, and to Professor V. F. Kolchin for useful discussions during the translation.

V. N. Sachkov

Introduction

With advances in cybernetics and closely related divisions of science, discrete mathematics has found increasing importance as a tool for the investigation of various models of functioning of technical devices and discretely operating systems.

A significant place in discrete mathematics is occupied by combinatorial methods whose applications can help in solving the problems of the existence and construction of arrangements of elements according to certain rules, and in the estimation of the number of such arrangements. Each arrangement determines a configuration which can be considered as a mapping of one set onto another with some restrictions posed by a particular problem. If the restrictions are complicated, then we are faced with the problem of determining conditions of existence and suggesting methods of construction of such configurations.

In Chapter 1 such questions are considered for block-designs and Latin squares. The results presented in the chapter are intended to provide an insight into the typical problems of this area of combinatorial mathematics.

Chapter 2 is devoted to transversals, usually referred to as systems of distinct representatives of a family of sets. Permanents are the main tools for calculating the number of transversals of a family of sets. In this chapter methods of calculating the values of permanents are considered. These calculations meet with more difficulties than do calculations of determinants, those objects which in respect of many other properties are close to permanents.

Much attention is given in the book to so-called enumerative problems of combinatorics. The solutions of such problems consist either of the suggestion of a method of search for combinatorial configurations of some class or in the evaluation of the number of the configurations, or both, in the search and evaluation.

In enumerative problems the main role is played by the generating function method, whose basic properties are given in Chapter 3. Depending on the problem in question the generating functions are considered either as elements of the ring of formal power series or as analytic functions of real or complex variables. If a generating function is an analytic function, we can find an asymptotic representation of the coefficients using the saddle point method, well-known in complex analysis for the estimation of contour integrals.

In Chapter 4, generating functions are used in enumerative problems of graph theory. A general method for constructing the generating functions enumerating the mappings of finite sets with some restrictions on the components is given. The enumeration of mappings with restrictions on contours and height provides a possibility for solving the corresponding enumerative problems for substitutions with restrictions on cycle lengths.

The rigorous definition of a combinatorial configuration requires formalization of the notion of the indistinguishability of elements. Currently, the approach to the introduction of the notion of indistinguishability in combinatorial analysis which uses the notion of equivalence classes generated by a group of substitutions G acting on the initial set of elements X is well known. In this approach the set X' of distinguishable elements coincides with the factor set with respect to a given equivalence relation. The problem of finding the number of distinct elements, which is usually called the enumerative problem, is reduced to the calculation of the coefficients of some polynomials depending on the residue classes of the group G. This method, known in combinatorics as Pólya's enumerative theory, began with the papers by Redfield (Redfield, 1927) and Pólya (Pólya, 1937) and was developed in de Bruijn's paper (de Bruijn, 1958). A presentation of this method and some of its applications are given in Chapter 6. It should be noted that Pólya enumerative theory is of great importance in many enumerative problems, especially in graph theory. However, in a number of important cases the application of the theory meets with considerable difficulties and proves to be inefficient. It is primarily concerned with well-known combinatorial schemes such as allocations, combinations, permutations with various additional properties, occupancy problems with restrictions on capacities of cells and different notions of distinguishability of elements, etc.

One of the central notions of this book is the so-called general combinatorial scheme introduced in Chapter 5. This scheme includes particular cases of almost all the known combinatorial models mentioned above. It allows us to formulate a unified method of construction of generating functions for special classes of enumerative combinatorial problems with some common properties. Note that some approaches suggested by Riordan in (Riordan, 1958) were used in the elaboration of this method.

The general combinatorial scheme provides a required level of mathematical rigor in formulations of various combinatorial problems and permits the unification of the methods of solution. It should be noted that in applying the general combinatorial scheme we achieve economy of presentation of material, since the necessity for separate presentation of particular combinatorial schemes no longer arises, as it usually does in monographs on combinatorial analysis.

A distinctive feature of this book is the large number of asymptotic formulae. In some cases, application of the asymptotic formulae can be considered as a tool for the simplification of the cumbersome explicit formulae obtained which cannot be used in calculations for large values of the parameters involved even if powerful computers are available. In other cases, an explicit formula cannot be obtained and the only alternative is an asymptotic formula obtained in some indirect way. In both types of case the efficiency of combinatorial methods in the absence of the use of the asymptotic formulae is problematical. At present, the use of asymptotic methods in combinatorial analysis receives wide recognition, but many interesting results are published in mathematical journals and have not yet been embodied in monographs on combinatorial analysis. It is hoped that this book will, to some extent, assist in closing this gap.

In conclusion, note that this book is not intended to present the most general results on the questions considered. It seems to me that it is more appropriate to focus our attention on typical results whose derivations well illustrate the combinatorial methods used, in order that the essence is not hidden by technical details.