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Probabilistic Methods in Combinatorial Analysis

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Probabilistic methods in combinatorial analysis

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Preface

During the last two decades a vast number of articles have been published in which probabilistic methods have been used successfully to solve combinatorial problems, and especially to obtain various asymptotic results concerning some or other characteristic of combinatorial objects. This book is aimed at readers interested in problems of this kind both from the theoretical point of view and from the point of view of possible applications.

The book may be used by students and postgraduates in combinatorics and other fields where asymptotic methods of probability theory are applied. In particular, the material contained in this book could be taught in courses on combinatorial structures such as graphs, trees and mappings with an emphasis on the asymptotic properties of their characterictics. We believe that the asymptotic results presented here provide specialists in probability theory with new examples of applications of general limit theorems.

This text assumes a standard graduate-level knowledge of probability theory and an aquaintance with typical facts drawn from a general introduction to functions of complex variables. For the reader's convenience, the relevant results of probability theory are briefly reviewed in Chapter 1. The preliminaries from combinatorial analysis are given in the introductions to each of the subsequent chapters. Readers who are interested in obtaining more detailed knowledge of the corresponding aspects of combinatorics are advised to study my book *'Combinatorial Methods in Discrete Mathematics'* or any other basic course devoted to this subject.

The individual chapters are independent to a certain extent and the reader interested in some particular problems will have no difficulty in their study if he or she turns directly to the corresponding sections. A list of the references mentioned in the text is given at the end of the book and may be used for the study of those combinatorial problems which could not be covered within the framework of the present text.

V.N. Sachkov

Preface to the English translation

The English translation differs slightly from the Russian edition of the book in the following. I have rewritten the Introduction, Subsections 2.2, 2.3 and 6.2.2. A number of minor changes have been made throughout the text to eliminate misprints and awkward proofs. Also, the list of references has been extended by the inclusion of articles and monographs devoted to relevant problems of probabilistic combinatorics which have appeared subsequent to publication of the Russian version of the book.

I am greatly indebted to Professor B. Bollobás and Professor V. Vatutin for their help and valuable advice during the preparation of the manuscript.

V.N. Sachkov

Introduction

Many branches of mathematics owe a debt to classical combinatorics. This is especially true of probability theory. Plenty of good examples show how combinatorial considerations lead to very deep and difficult probabilistic results. The links between combinatorial and probabilistic problems have played an important role in forming probability theory as a mathematical discipline, and now manifest themselves in elementary courses devoted to this subject. The initial stage of the development of probability theory was characterized by the essential contribution of combinatorial methods in forming the mathematical background of the science. The current situation is quite different: well-developed probabilistic methods find a wide range of applications in solving various combinatorial problems. This is revealed in the search for asymptotic results in combinatorial analysis, where the probabilistic formulations of combinatorial problems provide the possibility to use the working system of notions of probability theory effectively and to take advantage of the powerful techniques of limit theorems in finding asymptotic formulae. It is appropriate to mention here that asymptotic results play an essential role in combinatorial analysis: they simplify calculations in problems oriented to applications and present the whole picture of investigated phenomena in a more transparent form.

For convenience of references some basic notions and facts of probability theory are listed in the first chapter of the book. Although these facts are presented in a systematic and unified form, this part of the monograph is not assumed to be a substitute for a textbook on probability theory, but is directed to those readers who have some basic knowledge of the subject. Preference is given to those notions and theorems which will be used in subsequent chapters and thus collecting the material, we have paid particular attention to the discrete probability theory which has the closest links with combinatorial analysis. The first chapter also includes a number of results concerning continuous distributions which are used later in obtaining some asymptotic relations.

The main objects of investigation in subsequent chapters are nonnegative matrices, partitions of finite sets, mappings of finite sets and, in particular, permutations and graphs. We also consider equivalence classes specified on the sequences of a given length constituted by elements of partially ordered sets. These combinatorial structures include the so-called allocations of objects into cells, with various restrictions on the form of possible allocations that describe whether the objects and (or) cells are distinguishable or not. Each time, a probability distribution (as a rule, a uniform one) is specified

on the set of objects under consideration. The main problems in studying these objects are to find interrelations between the exact and asymptotic distributions of random variables associated with these objects. In essence, this is a way of solving combinatorial problems, and the well-established methods of combinatorial analysis are much applicable in finding the exact distribution. Probabilistic methods appear to be useful just at the stage of searching for asymptotic distributions.

Evidently, V.L. Goncharov was the first Russian mathematicians to become interested in problems of probabilistic combinatorics. In 1944 he published his paper 'Some facts from combinatorics', in which he used the method of generating functions to prove a number of limit theorems concerning series in random sequences and cycles in random permutations.

After this publication a large number of papers have appeared which study probability problems related to nonnegative matrices, partitions of sets, permutations and so on. The surveys [65, 104, 114] give a complete description of the contribution of the Russian mathematical school to the field of such problems. However, only a few monographs have been devoted to the subject. We hope that this book will fill the gap to some extent.

It should be noted that some of the results discussed in this book are given in simplified form. The reason of such simplifications is to facilitate readers the understanding of the essence of the used methods and to save them a possibility of independent estimation the permissibility of these or those generalizations. In difficult cases readers are advised to consult the literature cited at the end of the book.