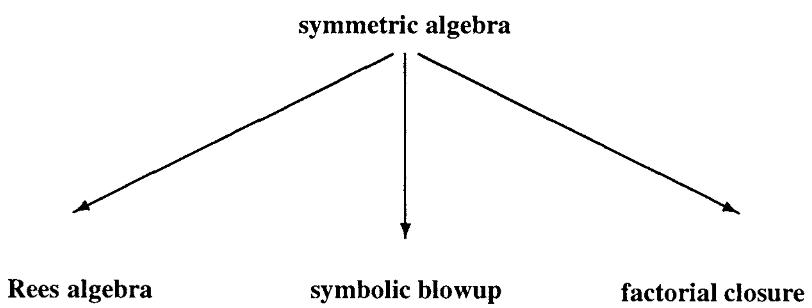
Introduction

There is a class of rings, collectively designated **blowup algebras**, that appear in many constructions in Commutative Algebra and Algebraic Geometry. They represent fibrations of a variety with fibers which are often affine spaces; a polynomial ring $R[T_1, \ldots, T_n]$ is the notorious example of such algebras. The algebras arise from module and categorical theoretic constructions over a base ring R, which has led to the terminology of algebras of linear type. Its uses include counterexamples to Hilbert's 14th Problem, the determination of the minimal number of equations needed to define algebraic varieties, the computation of some invariants of Lie groups, and several others. The main impetus for their systematic study has been the long list of beautiful Cohen–Macaulay algebras produced by the various processes. Finally, they provide a testing ground for several computational methods in Commutative Algebra.

Schematically these rings are:



They will be introduced individually, beginning with the most ubiquitous. A multiplicative $\mathbb{N}$–filtration $\mathcal{F}$, of a commutative ring R, is a sequence of subgroups $\{R_n,\ n \in \mathbb{N}\}$ of R with the property

$$R_m \cdot R_n \subset R_{m+n}.$$

The Rees algebra of $\mathcal{F}$ is the subring of the ring of polynomials $R[t]$

$$R(\mathcal{F}) := \sum_{n\in\mathbb{N}} R_n t^n.$$

If the filtration is decreasing, $R_{n+1} \subset R_n$, there are two other algebras attached to it: The extended Rees algebra $R(\mathcal{F})[t^{-1}]$, and the associated graded ring

$$\mathrm{gr}_{\mathcal{F}}(R) := \bigoplus_{n=0}^{\infty} R_n/R_{n+1}.$$

A major example is the I–adic filtration of an ideal I: $R_n = I^n$, $n \geq 0$. Its **Rees algebra**, which will be denoted by $\mathcal{R}(I)$ or $R[It]$, has its significance centered on the fact that it provides an algebraic realization for the classical notion of blowing–up a variety along a subvariety, and plays an important role in the birational study of algebraic varieties, particularly in the study of desingularization.

Although one of our purposes here is to study the algebraic properties of these rings, an equal effort will be placed on their ancestors, **symmetric algebras**, that has several other interesting descendants. Given a commutative ring R and an R–module E, the symmetric algebra of E is an R–algebra $S(E)$ which together with a R–module homomorphism

$$\pi : E \longrightarrow S(E)$$

solves the following universal problem. For a commutative R-algebra B and any R–module homomorphism $\varphi : E \longrightarrow B$, there exists a unique R–algebra homomorphism $\Phi : S(E) \longrightarrow B$ such that the diagram

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & B \\ \pi \downarrow & \nearrow \Phi & \\ S(E) & & \end{array}$$

is commutative. Thus, if E is a free module, $S(E)$ is a polynomial ring $R[T_1, \ldots, T_n]$, one variable for each element in a given basis of E. More generally, when E is given by the presentation

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \rightarrow 0, \quad \varphi = (a_{ij}),$$

its symmetric algebra is the quotient of the polynomial ring $R[T_1, \cdots, T_n]$ by the ideal $J(E)$ generated by the 1-forms

$$f_j = a_{1j}T_1 + \cdots + a_{nj}T_n, \ j = 1, \ldots, m.$$

Conversely, any quotient ring of a polynomial ring $R[T_1, \ldots, T_n]/J$, with J generated by 1–forms in the T_i's, is the symmetric algebra of a module. Like the classical blowup, the morphism

$$\mathrm{Spec}(S(E)) \longrightarrow \mathrm{Spec}(R)$$

is a fibration of $\mathrm{Spec}(R)$ by a family of hyperplanes. The case of a vector bundle, when E is a projective module, already warrants interest.

The other algebras are derived from $S(E)$ by effecting modifications on its components, some rather mild but others brutal. To show how this comes about, consider first the case of ideals. For an ideal $I \subset R$, there is a canonical surjection

$$\alpha : S(I) \rightarrow \mathcal{R}(I).$$

If, further, R is an integral domain, the kernel of α is just the R–torsion submodule of $S(I)$. This suggests the definition of the Rees algebra $\mathcal{R}(E)$ of an R–module as $S(E)/T$, with T the (prime) ideal of the R–torsion elements of $S(E)$.

Another filtration is that associated to the symbolic powers of the ideal I. If I is a prime ideal, its nth symbolic power is the I–primary component of I^n. (There is a more general definition if I is not prime.) Its Rees algebra

$$\mathcal{R}_s(I) := \sum_{n \geq 0} I^{(n)} t^n,$$

the **symbolic Rees algebra** of I, which also represents a blowup, inherits more readily the divisorial properties of R, but has its usefulness limited because it is not always Noetherian. The presence of Noetherianess in $\mathcal{R}_s(I)$ is loosely linked to the number of equations necessary to define set–theoretically the subvariety $V(I)$. In turn, the lack of Noetherianess of certain cases has been used to construct counterexamples to Hilbert's 14th Problem.

Wedged between the Rees algebra of an ideal I and its symbolic blowup lies the integral closure of $\mathcal{R}(I)$. It is also defined by a filtration but we shall be unable to deal with it other than algorithmically. On the other hand, there will be methods to ascertain whether a symmetric algebra is normal and to compute its divisor class group.

The last algebra attached to a module E is the graded bi–dual of $S(E)$

$$B(E) := \sum_{i \geq 0} (S_i(E))^{**},$$

where $S_i(E)^{**}$ is the R–module bi–dual of the ith graded component of $S(E)$. If R is a Noetherian factorial domain and E is finitely generated, then $B(E)$ is a factorial domain. It is called the **factorial closure** of $S(E)$.

A common thread of the three algebras derived from $S(E)$ is that each is obtained by the same process of taking the ring of global sections of Spec($S(E)$) on an appropriate affine open set.

To study the ideal theory of any class of algebras is, to a great extent, to make comparisons between two series of numbers: those arising from expressions of gross size—such as Krull dimension—and numbers measuring embeddings into regular (*i.e.* canonical) objects—depth is an example—and which have a more delicate geometric flavor. The running theme proper here is the ideal theory—Krull dimension, integrality, normality and factorization—of symmetric algebras, and its modifications. The aim is to highlight some of the known results, oftentimes technical points of their proofs, and to list significant open problems. At the same time, and this is emphasized, it seeks to provide gateways to other related developments in commutative algebra, but constraints of time and space leave this task barely started.

We shall now give an overall description of the contents. Brief introductions in each chapter have a more technical explanation of its results.

The first chapter focuses on the most primitive of the measures of an algebra: The analysis of the Krull dimension of a symmetric algebra $S(E)$. It is fairly complete in that there is an abstract general formula, and, based on it, a constructive method for computing the dimension in terms of a presentation of the module. It depends on the sizing of determinantal ideals, an exposition of which is also given. Two notions that will be visible throughout are introduced. First, we consider the family of conditions $\mathcal{F}_k$ on the Fitting ideals of a module. It will be used as a vehicle to read Krull dimension, and later of finer invariants such as divisor class groups. The other device is the technique of Jacobian duals, a process that attaches modules over different rings to certain ideals generated by quadrics. An important unresolved issue is when these special quadrics generate prime ideals.

The next two chapters deal with several generalizations of regular sequences, and there will be quite a diversity of them! A dominating theme is that of deciding when an ideal generated by linear forms in a partial set of variables is prime. Some interesting classes of examples are discussed in Chapter 2.

Chapter 3 constructs the **approximation complexes**, a family of chain complexes derived from Koszul complexes. They were introduced to make up for the lack of methods to find projective resolutions of the high symmetric powers of modules. Although they are not complexes of free modules, they permit one to ascertain the Cohen–Macaulayness of many blowup algebras. They work well when considerable information is know about the homology of the ordinary Koszul complexes is available. These complexes are the underpinnings for all the remaining exposition.

The next chapter is a primer on linkage theory, and highlights the connections between Koszul homology and residual intersections. It describes invariants of ideals in the linkage class of complete intersections, sketches out some differences between the odd and even linkage classes and in greater detail considers the Koszul homology of ideals associated to graphs. Hopefully it can be used as a hook to the beautiful work of Huneke and Ulrich on the structure of linkage and its multiple uses.

Chapter 5 has a fundamental character as it deals with normality of Rees algebras and basic constructions to generate new algebras from old ones, and effects a bridge between the special theory of ideals of linear type with more general ideals. It exploits the Noether like normalization expressed by a reduction of an ideal. Several recent results on the Hilbert function of primary ideals are explained away. New classes of Cohen–Macaulay rings are constructed.

There is a simple form of Serre's criterion, and an accompanying method to calculate the divisor class of normal algebras. It also explores some interesting relationships between subrings generated by monomials and their Rees algebras.There are a number of elementary techniques to produce Rees algebras with predictable properties from simpler algebras, along with computational tests of normality. Finally it introduces a novel technique (we almost said, technology) that seeks to apply the methods used to study algebras of linear type to much more general Rees algebras.

The next focus is the canonical module of a symmetric algebra. It occurs prominently in an unsettled conjecture purporting to describe all such algebras which are factorial: that they are complete intersections. Some cases are affirmed, and they can be viewed as assertions of homological rigidity. Other related results include the proof of the *Zariski–Lipman conjecture* for symmetric algebras over polynomial rings. It ends with a detailed examination of the structure of the canonical module.

Chapters 7 aims at finding ideal transforms of symmetric algebras. This includes the factorial closure of these algebras and the computation of symbolic blowups. A counterexample by Roberts to Hilbert's 14th Problem can be framed in this context. On the other hand, modules with linear presentation provide a vigorous set of fresh problems.

The next chapter seeks ways to determine the equations of the Rees algebras of ideals that are not necessarily of linear type. It is heavily mediated by homological algebra. There are several *ad hoc* approaches including computer–assisted methods. (There is even a little proof by computer!)

Chapter 9 is about commuting varieties of algebras, particularly Lie algebras. It is focused on two things: proving that certain ideals generated by quadrics are prime and bringing some invariant theory into the picture. It is very promising in applications, or at least looks so to this non–expert.

The last chapter discusses computer algebra methods in Commutative Algebra. It introduces the Buchberger's algorithm and traces out its role in the practice of computation in polynomial ideal theory. Because almost any application of these algorithms seems to lie on the brink of combinatorial explosion, greater emphasis is put on what might be called second generation methods, whereby theoretical information has to be fed into the computation, in order to beat the beast of complexity. (As it will be seen, they are still crude artifacts.) Among these methods are: Noether normalization, various forms of the *Nullstellensatz*, elements of primary decomposition, primality testing, computation of the integral closure of affine domains and of ideal transforms. There is another application domain for some of these techniques, the development of algebraic *solvers*: Programs to compute the solutions of systems of polynomial equations. Because of their need for *practicality* they tend to be integrated with numerical routines.

There are a great number of related topics that were not treated. The bountiful bibliography is intended to make up for that fault. Actually, at the time of this writing, several developments are leading to a very deep understanding of which Rees algebras are Cohen–Macaulay. Regretfully these will go unreported.

Chapter 1

Krull Dimension

This chapter has for aim the development of techniques to determine the most basic measure of a symmetric algebra—its Krull dimension. It requires an exposition of the classical Fitting ideals of modules, with estimates of codimensions of determinantal ideals.

The Krull dimension of a symmetric algebra $S(E)$ of a module E turns out to be connected to the invariant $b(E)$ of the module E introduced by Forster [71], a quarter of a century ago, that bounds the number of generators of E. This was shown by Huneke and Rossi [150]; furthermore, it was accomplished in a manner that makes the search for dimension formulas for $S(E)$ much easier. Based on slightly different ideas, [261] gives another proof of that result in terms of the heights of the Fitting ideals of E. It makes for an often effective way of determining the Krull dimension of $S(E)$.

The Fitting ideals are introduced from various directions, and their divisorial properties are noted. It is to be expected that these ideals play such an important role since they code the symmetric algebra of the module. In later chapters, their primary components will be used in expressing the divisor class group of normal algebras.

One topic which is central to this study, that of ascertaining when a symmetric algebra is an integral domain will have a very brief development here but will recur under various circumstances in several other chapters. One has been unable to capture the general properties that lead to symmetric algebras whose prime spectrum is irreducible.

At the end of the chapter we introduce a formalism—*the Jacobian dual*—that permits viewing some ideals generated by quadrics as ideals of definition of two distinct modules over different rings. It has proven itself to be an useful technique.

1.1 Fitting Ideals

The Fitting ideals of a module E generalize the classical invariants of finitely generated modules over Dedekind domains. In general, they do not fully determine the module. Nevertheless they convey many of its properties and are literally read off the presentation of the module.

Determinantal ideals

Given a presentation of an R–module E

$$R^m \xrightarrow{\varphi} R^n \longrightarrow E \longrightarrow 0,$$

its Fitting invariants are the various ideals generated by the minors of a matrix representation of φ.

We recall some notation in the following twin definitions:

Definition 1.1.1 Given a $n \times m$ matrix φ with entries in the commutative ring R, we set $I_t(\varphi)$ for the ideal generated by the $t \times t$ minors of the matrix φ. The ideal $I_1(\varphi)$ will be called the content of φ.

Definition 1.1.2 For an integer $0 \leq r < n$ the ideal $f_r(E)$ generated by the minors of order $n - r$ of the matrix φ is the rth Fitting ideal of E. If $r \geq n$ one puts $f_r(E) = R$.

The shift in the order of the minors makes the definition independent of the presentation of E; this was already shown by Fitting [69]. It was an early form of what became known as the Schanuel's lemma of Homological Algebra. When working from a fixed presentation of the module dealing with the ideals $I_t(\varphi)$ will be more convenient than using the notation of the Fitting ideals.

Another way of defining $f_r(E)$ is through the following broader notion.

Definition 1.1.3 Let $\varphi: F \longrightarrow G$ be a homomorphism of R–modules. The *order ideal* of φ is the ideal

$$o(\varphi) = \sum f(\varphi(F)), \ f \in \mathrm{Hom}_R(G, R).$$

One could then put the notion above in the form

$$f_r(E) = o(\wedge^{n-r}\varphi)$$

where $\wedge^{n-r}\varphi$ denotes the $(n-r)$th exterior power of the presentation matrix of E. The advantage of the previous definition is its accessibility. In fact, when working out from a fixed presentation of E, we shall often state conditions directly in terms of the $I_t(\varphi)$.

Remark 1.1.4 If $\psi: R \longrightarrow S$ is a ring homomorphism and E is a finitely generated module, one has from the definition that $f_r(E \otimes_R S) = \psi(f_r(E)) \cdot S$.

Elementary properties

We shall later be concerned with estimates of the heights of the Fitting ideals. Due to their lengthy expression, it is convenient to have other ways of describing them at least up to radicals. We briefly discuss two of them.

First observe that if $a_1(E)$ denotes the annihilator of E then

$$a_1(E)^n \subset f_0(E) \subset a_1(E).$$

More generally, the *invariant factors* of E are

$$a_r(E) = a_1(\wedge^r E),\ r \geq 1.$$

In addition, one also has the *Kaplansky invariants* of E: If $\mathcal{E}_r$ denotes the set of submodules of E that can be generated by r elements, then

$$k_r(E) = \sum a_1(E/F),\ F \in \mathcal{E}_r.$$

The relationship between them is summed up in:

Proposition 1.1.5 *Let E be a finitely generated R–module. Then*

$$\sqrt{f_r(E)} = \sqrt{k_r(E)} = \sqrt{a_{r+1}(E)}.$$

Proof. For a prime ideal P of R, let E_P be the localization of E at P. Denote by $\nu(E_P)$ the minimal number of generators of E_P. It follows easily that

$$\nu(\wedge^r E_P) = \binom{\nu(E_P)}{r}.$$

We then have:

(i) $a_r(E) \subset P$ if and only if $\nu(E_P) \geq r$.

(ii) If $k_r(E) \not\subset P$, there is $x \in k_r(E) \setminus P$ such that $x \cdot (E/F) = 0$, for some submodule $F \in \mathcal{E}_r$. Localizing at P one has $E_P = F_P$ and $\nu(E_P) \leq r$. Conversely, if $\nu(E_P) \leq r$ there is a submodule F generated by r elements and $y \notin P, y \cdot E \subset F$. Thus

$$k_r(E) \subset P \text{ if and only if } \nu(E_P) > r \text{ and } \sqrt{k_r(E)} = \sqrt{a_{r+1}(E)}.$$

(iii) Finally, assume $f_r(E) \not\subset P$. Choosing a minimal presentation of E_P this means $\nu(E_P) \leq r$. Since the converse is clear, altogether we have that $f_r(E) \subset P$ if and only if $\nu(E_P) \geq r+1$. □

Divisors and factoriality

A classical application of these invariants is in developing a theory of divisors, that is, an association of torsion modules with a class of ideals with good multiplicative properties.

To simplify the discussion, let us assume that R is an integral domain (or that, at appropriate places, ideals contain regular elements). First, one singles out the following set of prime ideals (*cf.* [71]). Let $\mathcal{P}$ be the collection of prime ideals that are minimal over an ideal of the type $(a):_R b = \{r \in R \mid rb \in (a)\}$. In other words, $P \in \mathcal{P}$ if and only if R_P has depth one.

Some of the properties of $\mathcal{P}$ are the following. A fractionary ideal I of R is a submodule of the field of fractions K of R for which there exists $0 \neq x \in R$, such that $xI \subset R$. The set $\{x \in K \mid xI \subset R\}$ will be denoted by I^{-1}. The ideal I is called *reflexive* or *divisorial* if $(I^{-1})^{-1} = I$, that is, I is reflexive as an R–module.

Proposition 1.1.6 *Let I be a fractionary ideal of R.*

(a) *If I is not contained in any $P \in \mathcal{P}$, $I^{-1} = R$.*

(b) *I is a reflexive ideal if and only if $I = \bigcap_{P\in\mathcal{P}} I_P$.*

Proof. It is an easy exercise that is left to the reader. □

There is a composition on the set $Div(R)$ of all divisorial ideals of R. For two divisorial ideals I and J define

$$I \circ J = ((I \cdot J)^{-1})^{-1}.$$

The submonoid formed by the invertible ideals will be denoted by $Inv(R)$. They compose, with other elements of $Div(R)$, through ordinary multiplication of ideals.

We come to one of our aims, the definition of a function from torsion modules of finite projective dimension into $Inv(R)$.

Definition 1.1.7 Let

$$F_1 \longrightarrow F_0 \longrightarrow E \longrightarrow 0$$

be a projective presentation of the torsion module E; put $I = f_0(E)$. The ideal $\mathbf{d}(E) = (I^{-1})^{-1}$ is the *determinantal divisor* of E.

On modules of finite projective dimension the function $\mathbf{d}(\cdot)$ has the following remarkable property ([190]):

Theorem 1.1.8 *Let E be a torsion module of finite projective dimension. Then $\mathbf{d}(E)$ is an invertible ideal.*

Lemma 1.1.9 *If* proj dim $(E) = 1$, *then* $\mathbf{d}(E)$ *is an invertible ideal of R.*

Proof. We prove more generally the following statement. Let E be a finitely generated module over a commutative ring R, annihilated by a regular element. Then $\text{proj dim}_R(E) \leq 1$ if and only if $f_0(E)$ is an invertible ideal of R. For that, let

$$0 \longrightarrow G \xrightarrow{\varphi} F \longrightarrow E \longrightarrow 0$$

be exact with F a finitely generated free module. If $\text{proj dim}_R(E) \leq 1$, G is a finitely generated projective module that locally has the same rank as F. This means that locally $f_0(E)$ is generated by the determinant of φ.

For the converse one may assume, after localization at a prime ideal, that $f_0(E)$ is given by an specific minor of φ. It is then easy to see that G is generated by the corresponding column vectors that enter in the minor (see [190] for additional details). □

Lemma 1.1.10 $\mathbf{d}(\cdot)$ *is an additive function on the category of torsion modules which have finite projective dimension when localized at the primes in $\mathcal{P}$. That is, if*

$$0 \to E \longrightarrow F \longrightarrow G \to 0$$

is an exact sequence of such modules then

$$\mathbf{d}(E) \circ \mathbf{d}(G) = \mathbf{d}(F).$$

Proof. Construct a projective presentation of the exact sequence

$$\begin{array}{ccccccccc}
0 & \to & E_1 & \longrightarrow & F_1 & \longrightarrow & G_1 & \to & 0 \\
 & & \alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \\
0 & \to & E_0 & \longrightarrow & F_0 & \longrightarrow & G_0 & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & E & \longrightarrow & F & \longrightarrow & G & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

The matrix β is given in terms of the others by

$$\begin{bmatrix} \alpha & \star \\ 0 & \gamma \end{bmatrix}.$$

To show the assertion it is enough to verify it at the prime ideals of $\mathcal{P}$, but then α and γ are square matrices and $\det(\beta) = \det(\alpha) \cdot \det(\gamma)$. □

Proof of Theorem. We may assume that $\text{proj dim}_R(E) > 1$. Let x be a regular element in the annihilator of E. If

$$F \longrightarrow E \to 0$$

is a presentation of E with F free, consider the exact sequence

$$0 \to G \longrightarrow F/xF \longrightarrow E \to 0.$$