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Introduction

1.1. Sperner's theorem

We start our investigations with the theorem that was the cornerstone for the whole theory. In the thirties, forties, and fifties few further results of a similar kind were published. But beginning with the sixties, the combinatorics of finite sets has undergone spectacular growth. Not only have subsets of a finite set been studied, but also more general objects like partially ordered sets. Many important results in this area can be found in this book.

***Theorem 1.1.1* (*Sperner* [436]).** *Let n be a positive integer and $\mathcal{F}$ be a family of subsets of $[n] := \{1, \dots, n\}$ such that no member of $\mathcal{F}$ is included in another member of $\mathcal{F}$, that is, for all $X, Y \in \mathcal{F}$ we have $X \not\subseteq Y$. Then*

(*a*)

$$|\mathcal{F}| \leq \begin{cases} \binom{n}{\frac{n}{2}} & \text{if } n \text{ is even,} \\ \binom{n}{\frac{n-1}{2}} & \text{if } n \text{ is odd.} \end{cases}$$

(*b*) *Equality holds iff*

$$\mathcal{F} = \begin{cases} \{X \subseteq [n] : |X| = \frac{n}{2}\} & \text{if } n \text{ even,} \\ \{X \subseteq [n] : |X| = \frac{n-1}{2}\} \text{ or } \{X \subseteq [n] : |X| = \frac{n+1}{2}\} & \text{if } n \text{ odd.} \end{cases}$$

Proof. The following presents Sperner's original approach. Clearly the families given in (b) satisfy the conditions of the theorem and have the corresponding size. Hence we must show that there do not exist "better" (resp. "other") families. Let

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$\mathcal{F}$ be any family of maximum size satisfying the conditions of the theorem. Let

$$l(\mathcal{F}) := \min\{i : \text{ there is some } X \in \mathcal{F} \text{ with } |X| = i\},$$

$$u(\mathcal{F}) := \max\{i : \text{ there is some } X \in \mathcal{F} \text{ with } |X| = i\}.$$

For brevity we write l instead of $l(\mathcal{F})$ if $\mathcal{F}$ is clear from the context. Let

$$\mathcal{G} := \{X \in \mathcal{F} : |X| = l\},$$

$$\mathcal{H} := \{Y \subseteq [n] : |Y| = l + 1 \text{ and there is some } X \in \mathcal{G} \text{ with } X \subset Y\},$$

$$\mathcal{F}' := (\mathcal{F} - \mathcal{G}) \cup \mathcal{H}.$$

Claim 1. The family $\mathcal{F}'$ satisfies the conditions of the theorem.

Proof of Claim 1. The only obstacle could be the existence of some $Y \in \mathcal{H}$ and some $Z \in \mathcal{F} - \mathcal{G}$ such that $Y \subset Z$. But by definition of $\mathcal{H}$ we would find also some $X \in \mathcal{G} \subseteq \mathcal{F}$ with $X \subset Y$. Thus $X \subset Z$, contradicting the fact that $\mathcal{F}$ satisfies the conditions of the theorem. □

Claim 2. Let $l \leq \frac{n-1}{2}$. Then $|\mathcal{F}'| \geq |\mathcal{F}|$, and $|\mathcal{F}'| = |\mathcal{F}|$ implies $l = \frac{n-1}{2}$.

Proof of Claim 2. Let us count the number N of pairs (X, Y) with $X \in \mathcal{G}$, $Y \in \mathcal{H}$, $X \subset Y$ in two different ways. For a fixed member X of $\mathcal{G}$, we can find exactly $n - l$ corresponding sets Y since Y can be obtained in a unique way from X by adding one element of $[n] - X$. Thus

$$N = |\mathcal{G}|(n - l). \tag{1.1}$$

For a fixed member Y of $\mathcal{H}$, we can find analogously $l + 1$ sets X with $X \subset Y$, $|X| = l$. But it is not necessary that all these sets X belong to $\mathcal{G}$. Thus

$$N \leq |\mathcal{H}|(l + 1). \tag{1.2}$$

By (1.1) and (1.2) and because of $l \leq \frac{n-1}{2}$,

$$|\mathcal{G}|(n - l) \leq |\mathcal{H}|(l + 1), \tag{1.3}$$

$$\frac{|\mathcal{H}|}{|\mathcal{G}|} \geq \frac{n - l}{l + 1} \geq \frac{n - \frac{n-1}{2}}{\frac{n-1}{2} + 1} = 1,$$

where the last inequality is only an equality if $l = \frac{n-1}{2}$. Since $\mathcal{F}$ satisfies the conditions of the theorem, $\mathcal{F} \cap \mathcal{H} = \emptyset$. Thus

$$|\mathcal{F}'| = |\mathcal{F}| - |\mathcal{G}| + |\mathcal{H}| \geq |\mathcal{F}| \quad \left(\text{equality implies } l = \frac{n-1}{2}\right).$$

□

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Recall that we have already chosen $\mathcal{F}$ as a family of maximum size that satisfies the conditions of the theorem. We obtain from Claims 1 and 2

$$l(\mathcal{F}) \geq \frac{n-1}{2} \text{ and (analogously) } u(\mathcal{F}) \leq \frac{n+1}{2}$$

because otherwise we could construct a family $\mathcal{F}'$ of larger size. If n is even, we are already done. So let n be odd. If $l(\mathcal{F}) = u(\mathcal{F})$, both values are either $\frac{n-1}{2}$ or $\frac{n+1}{2}$ and, consequently,

$$|\mathcal{F}| \leq \binom{n}{\frac{n-1}{2}} = \binom{n}{\frac{n+1}{2}}. \tag{1.4}$$

Thus assume that $l(\mathcal{F}) = \frac{n-1}{2}$, $u(\mathcal{F}) = \frac{n+1}{2}$. In this case we will obtain a contradiction. By Claim 2,

$$|\mathcal{F}| \leq |\mathcal{F}'| \leq \binom{n}{\frac{n+1}{2}}. \tag{1.5}$$

Since $\mathcal{F}$ is of maximum size, we must have equality in (1.5), thus also in (1.2), and this is possible only if for every $Y \in \mathcal{H}$ each l-element subset of Y belongs to $\mathcal{G}$. But consider under all pairs (Y, Z) with $Y \in \mathcal{H}$, $Z \in \mathcal{F} - \mathcal{G}$ such a pair for which $|Y \cap Z|$ is maximum. Since $|Y| = |Z| = l + 1$, $Y \neq Z$, there exist some $y \in Y - Z$ and some $z \in Z - Y$. In view of the preceding remarks, $Y - \{y\}$ must belong to $\mathcal{G}$; thus $Y' := (Y - \{y\}) \cup \{z\}$ belongs to $\mathcal{H}$. Now $|Y' \cap Z| = |Y \cap Z| + 1$ is a contradiction to the maximality of $|Y \cap Z|$. ■

This result (or at least part (a)) was obtained independently by several other mathematicians. As examples we mention here Gilbert [223] and the succeeding paper of Mikheev [369]. Using

$$\left\lfloor \frac{n}{2} \right\rfloor := \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n-1}{2} & \text{if } n \text{ is odd,} \end{cases}$$

part (a) of Theorem 1.1.1 reads: $|\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Sperner's theorem can be always applied if one works with families of subsets that are pairwise incomparable with respect to inclusion. Here we consider only one example. It was found by Demetrovics [130] in a study of the relational model of data structures proposed by Codd [118] and Armstrong [33]. Suppose we are given m persons $P_1, \ldots, P_m$ and n attributes $A_1, \ldots, A_n$ like last name, first name, date of birth, place of birth, weight, and so forth. For each person, each attribute takes on a unique value. Using a right coding, we may suppose that each such value is a natural number. Thus all data on the persons can be represented by an $m \times n$-matrix $D = (d_{ij})$, where d_{ij} is the value of A_j for person P_i. We say that a set of attributes $\{A_j : j \in X\}$, and, briefly, the set $X \subseteq [n]$, is a *key* if for fixed values of A_j, $j \in X$, there exists at most one person P_i that has these values – that

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is, if

$$d_{ij} = d_{i'j} \text{ for all } j \in X \text{ implies } i = i'$$

(one already "knows" the person if one knows his values of the attributes of a key). A key $X \subseteq [n]$ is called a *minimal key* if there is no key X' with $X' \subset X$.

Corollary 1.1.1. *For n attributes the number of minimal keys is not greater than $\binom{n}{\lfloor \frac{n}{2} \rfloor}$, and this bound is the best possible.*

Proof. It is trivial to see that the family $\mathcal{F}$ of *minimal* keys satisfies the conditions of Theorem 1.1.1, which proves the upper bound. To see that this bound can be attained, we must construct a corresponding matrix D. We do this here in a simple way with a large m, namely $m := \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + 1$. Set all entries of the first row of D equal to 1. Then order all n-dimensional rows with $\lfloor \frac{n}{2} \rfloor - 1$ ones and $n + 1 - \lfloor \frac{n}{2} \rfloor$ zeros in any way, but count them from 2 up to $\binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + 1$. Define the ith row of D to be the ith row from above, but with all zeros replaced by the number i, $i = 2, \ldots, \binom{n}{\lfloor \frac{n}{2} \rfloor - 1} + 1$. Then every $\lfloor \frac{n}{2} \rfloor$-element subset of $[n]$ is a key since either we find in the corresponding places only ones – and this can be the case only in the first row, or we find some number $i \neq 1$ – and this can be the case only in the ith row. Moreover, it is easy to see that there is no key of size smaller than $\lfloor \frac{n}{2} \rfloor$; thus we have indeed $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ minimal keys. ■

The preceding construction is due to Demetrovics and Katona [131]. For more information on similar combinatorial problems of data structures, see, for example, Demetrovics and Katona [131] and Demetrovics and Son [132].

1.2. Notation and terminology

The main objects considered in this book are *partially ordered sets* (abbreviated as *posets*), which are sets equipped with a reflexive, antisymmetric, and transitive relation (order relation). Throughout we suppose that the posets are *finite*. For the sake of brevity we will not distinguish between the poset and the underlying set. For two *comparable* (i.e., related) elements p, q of a poset P, we write in the usual way $p \leq q$ or, equivalently, $q \geq p$. Two posets P and Q are called *isomorphic* (denoted by $P \cong Q$) if there is a bijective mapping φ from P onto Q (called *isomorphism*) such that $p \leq q$ iff (i.e., if and only if) $\varphi(p) \leq \varphi(q)$. An *automorphism* of P is an isomorphism from P onto P.

Sometimes we study more general objects, namely graphs. An (*undirected, simple*) *graph* $G = (V, E)$ is a set V, called the *vertex set* or *point set*, together with a set E of two-element subsets of V, called the *edge set*. The *degree* $d(v)$ of a vertex $v \in V$ is defined as the number of edges containing v. The graph is called *regular of degree* d if $d(v) = d$ for all $v \in V$. We speak of *directed*

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graphs (*digraphs*) $G = (V, E)$ if E consists of (ordered) pairs (p, q) of different elements p, q of V and we call the elements of E *arcs*. For $e = \{p, q\}$ (p, q are the *endpoints* of e) (resp. $e = (p, q)$, p is the *starting point*, q is the *endpoint*), we write briefly pq, and in the directed case we use the notation $e^- := p, e^+ := q$. Moreover in the directed case we allow more than one arc between points p and q; thus E is a *multiset* of arcs.

The element q of a poset P is said to *cover* the element p (denoted by $p \lessdot q$ and $q \gtrdot p$) if $q > p$ and if $q \geq q' > p$ implies $q = q'$. Obviously, the order relation is the reflexive and transitive closure of the cover relation. A poset P can be illustrated by its *Hasse diagram*, which is a digraph $H(P) = (P, E(P))$ whose vertex set is P and whose arc set $E(P)$ consists of all pairs (p, q), where $p \lessdot q$. In figures we always have $q \gtrdot p$ if p and q are joined by a straight line and q lies higher than p. The *Hasse graph* is the underlying undirected graph of the Hasse diagram. An element p of P is called *minimal* (*maximal*) if $q \leq p$ ($q \geq p$) implies $q = p$. For two elements p, q of P, we define the *interval* $[p, q]$ to be the set of all elements of P lying between p and q; that is, $[p, q] = \{v \in P : p \leq v \leq q\}$.

A subset of pairwise comparable elements of a poset P is said to be a *chain*. We denote chains by $C = (c_0 < \cdots < c_h)$, which gives us not only the elements but also the relation between them. The number h is called the *length of* C. The *height function* assigns with each element of P the length of a longest chain with p at the top. A chain is called *saturated* if it has the form $C = (c_0 \lessdot \cdots \lessdot c_h)$, and it is called *maximal* if, in addition, c_0 and c_h are minimal and maximal elements of P, respectively.

An *antichain* is a subset of pairwise incomparable elements of P. Subsets of a poset will often be called *families* too (motivated by families of subsets of a set). Antichains are also called *Sperner families*. A *k-family* is a family in P containing no chain of $k+1$ elements in P, thus a 1-family is an antichain. Usually we denote families by roman letters F, G, and so on. If P is the Boolean lattice (to be defined in the next section) or if P is very similar to the Boolean lattice we also use script letters $\mathcal{F}$, $\mathcal{G}$, and so forth.

We speak of *maximal families* and *maximum families* satisfying various conditions. "Maximal" means not contained in any other; "maximum" means maximum-sized.

For graphs $G = (V, E)$, we define a subset C of V to be a *clique* if any two elements of C are joined by an edge (i.e., are *adjacent*), and a subset I of V is called *independent* if no two elements of I are adjacent. A *matching* in a graph $G = (V, E)$ is a subset M of E of pairwise *nonadjacent* edges; that is, no two edges of M have a common endpoint.

We often consider extremal problems not in a poset but in a *weighted poset* (P, w), which is a poset P together with a function (called a *weight function*) w from P into the set $\mathbb{R}_+$ of nonnegative real numbers. If $w(p) > 0$ for all $p \in P$, then (P, w) is called a *positively weighted poset*. The *weight* $w(F)$ of a family F of

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(P, w) is defined by $w(F) := \sum_{p\in F} w(p)$. Every poset P can be considered as a weighted poset (P, w) where $w \equiv 1$; that is, $w(p) = 1$ for all $p \in P$. We identify P and $(P, 1)$. The maximum weights of an antichain and a k-family in (P, w) are denoted by $d(P, w)$ and $d_k(P, w)$, respectively. The parameter $d(P, w)$ is called the *width* of (P, w).

Given a weighted poset (P, w) and a subset F of P, the poset whose underlying set is F and whose elements are ordered and weighted as in (P, w) is called the *poset induced by* F. The *dual* (P^*, w) of (P, w) has the same underlying set and the same weight function as (P, w), but it is ordered by $p \leq_{P^*} q$ iff $p \geq_P q$.

The (*direct*) *product* $P \times Q$ *of the posets* P *and* Q is defined to be the set of all pairs (p, q), $p \in P, q \in Q$, with the order given by $(p, q) \leq_{P\times Q} (p', q')$ iff $p \leq_P p'$ and $q \leq_Q q'$. Moreover, the *product* $(P, v) \times (Q, w)$ *of the weighted posets* (P, v) *and* (Q, w) is the product of P and Q together with the weight function $v \times w$ defined by $(v \times w)(p, q) := v(p)w(q)$, $p \in P, q \in Q$. We denote a product of n copies of (P, w) by $(P, w)^n$, and for $(P_1, w_1) \times \cdots \times (P_n, w_n)$ we write briefly $\prod_{i=1}^n (P_i, w_i)$.

Given a group G of automorphisms of a poset P, a nonempty subset A of P is called an *orbit* if for all $p, q \in A$ there is some $\varphi \in G$ such that $\varphi(p) = q$ and if A is maximal with respect to this property. It is easy to see that the union of all orbits is a partition of P. Now the *quotient of* P *under* G (denoted by P/G) is the poset of all orbits ordered in the following way: $A \leq_{P/G} B$ iff there are some $a \in A, b \in B$ such that $a \leq_P b$ (it is easy to see that P/G is really a poset). The *weighted quotient* is the quotient together with the weight function w/G defined by $w/G(A) := |A|$, $A \in P/G$.

Given two posets P and Q, a mapping $\varphi : P \to Q$ is called *order preserving* if $p \leq q$ implies $\varphi(p) \leq \varphi(q)$. If Q is the set $\mathbb{R}$ with the natural ordering, we speak of *increasing functions*. *Decreasing functions* are defined in an analogous way. The *characteristic function* of a subset S of P is defined and denoted by

$$\varphi_S(p) := \begin{cases} 1 & \text{if } p \in S, \\ 0 & \text{otherwise.} \end{cases}$$

The *support* of a function $f : P \to \mathbb{R}$ is the set $\text{supp}(f) := \{p \in P : f(p) \neq 0\}$.

A subset F of a poset P is called a *filter* (*ideal*) if $p \in F$ and $q \geq p$ ($q \leq p$) imply $q \in F$. Sometimes filters (ideals) are also called *upper ideals* (*lower ideals*). A filter (ideal) F is said to be *generated by a subset* S *of* P if $F = \{p \in P : p \geq q \ (p \leq q) \text{ for some } q \in S\}$. If S contains only one element we speak of *principal* filters and ideals.

Given two elements p, q of P, the element v is called *supremum* (*infimum*) of p and q – denoted by $v = p \vee q$ ($v = p \wedge q$) – if $v \geq p, v \geq q$ and if $w \geq p, w \geq q$ imply $w \geq v$ (if $v \leq p, v \leq q$ and if $w \leq p, w \leq q$ imply $w \leq v$). In an analogous way we define the supremum (infimum) of any subset A of P, which

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we denote by $\sup A$ ($\inf A$). Most of the examples considered in this book are *lattices*, that is, posets P in which $p \vee q$ and $p \wedge q$ exist for all $p, q \in P$.

Further, almost all posets that we will study are *ranked posets*, that is, posets together with a rank function. Here a *rank function* of a poset P is a function r from P into the set $\mathbb{N}$ of all natural numbers such that $r(p) = 0$ for *some* minimal element p of P and $p \lessdot q$ implies $r(q) = r(p) + 1$. Note that we do not suppose – as traditionally – that $r(p) = 0$ for all minimal elements p of P. If in a ranked poset every minimal element has rank 0 and every maximal element has the same rank, we speak of a *graded poset* (note that in any poset there is at most one rank function with this property). Given a ranked poset P, r_P denotes throughout its rank function, but generally we omit the index P and merely write r. The number $r(P) := \max\{r(p) : p \in P\}$ is called the *rank of* P (note the difference from the weight $w(P)$ of a weighted poset (P, w), which we defined by $w(P) := \sum_{p \in P} w(p)$). Very often we set for the sake of brevity $n := r(P)$.

A subset F of a graded lattice is called *t-intersecting* (*t-cointersecting*) if $r(p \wedge q) \geq t$ ($r(p \vee q) \leq r(P) - t$) for all $p, q \in F$. *Intersecting* (*cointersecting*) is an abbreviation for 1-intersecting (1-cointersecting).

The *dual of a ranked poset* P is the dual P^* of P together with the rank function $r_{P^*} := r_P(P) - r_P(p)$ for all $p \in P$. Moreover, the *product of two ranked posets* P, Q is defined to be the poset $P \times Q$ together with the rank function $r_{P \times Q}$ given by $r_{P \times Q}(p, q) := r_P(p) + r_Q(q)$. For a ranked poset P, we define the *ith level* by $N_i(P) := \{p \in P : r(p) = i\}$; its size $W_i(P) := |N_i(P)|$ is called the *ith Whitney number*, $i = 0, \ldots, r(P)$ (when there is no danger of ambiguity, we write briefly N_i and W_i). It is useful to define $N_i := \emptyset$ and $W_i := 0$ if $i \notin \{0, \ldots, r(P)\}$. Obviously, each level of a ranked poset is an antichain, and the union of k levels is a k-family. The *rank-generating function* $F(P; x)$ of a ranked poset is defined by $F(P; x) := \sum_{p \in P} x^{r(p)} \left(= \sum_{i=0}^{r(P)} W_i x^i\right)$. It is easy to see that $F(P \times Q; x) = F(P; x)F(Q; x)$ if P and Q are ranked. For $S \subseteq \{0, \ldots, r(P)\}$, we define the *$S$-rank-selected subposet* (P_S, w_S) as the subposet induced by $P_S := \{p \in P : r(p) \in S\}$ together with the induced weights w_S.

For ranked posets P, Q of the same rank, we define the *rankwise* (*direct*) *product* $P \times_r Q$ to be the set $\cup_{i=0}^{r(P)} N_i(P) \times N_i(Q)$ together with the relation $(p, q) \leq_{P \times_r Q} (p', q')$ if $p \leq_P p'$ and $q \leq_Q q'$. If we have, in addition, weights v and w on P and Q, resp., then, as for usual products, $(v \times_r w)(p, q) := v(p)w(q)$.

Given a family F in a ranked poset P, the set of rank i elements of F is denoted by F_i, and the numbers $f_i := |F_i|$ are called *parameters of* F, $i = 0, \ldots, r(P)$. The vector $\boldsymbol{f} = (f_0, \ldots, f_{r(P)})^{\mathrm{T}}$ is called the *profile of* F. If $F = F_i$ for some i then we call F *i-uniform*.

For an element p of P, we define and denote the *upper* (resp. *lower*) *shadow of* p by $\nabla(p) := \{q \in P : q \gtrdot p\}$ (resp. $\Delta(p) := \{q \in P : q \lessdot p\}$). More generally, if P is ranked, let the *upper* (resp. *lower*) *k-shadow of* p be defined and denoted by $\nabla_{\to k}(p) := \{q \in N_k : q \geq p\}$ (resp. $\Delta_{\to k}(p) := \{q \in N_k : q \leq p\}$).

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The $(k-)$ *shadows of a subset of* P are the unions of the $(k-)$ shadows of its elements.

More generally, given a weighted and ranked poset (P, w), the weight $w(N_i)$ of the ith level N_i of P is called the *weighted ith Whitney number*. We mostly use the following definitions in the $w \equiv 1$ case (where $W_i = w(N_i)$). The weighted and ranked poset (P, w) is said to have the *k-Sperner property* if the maximum weight of a k-family in (P, w) equals the largest sum of k weighted Whitney numbers in (P, w), that is, if

$$d_k(P, w) = \max\{w(N_{i_1}) + \cdots + w(N_{i_k}) : 0 \leq i_1 < \cdots < i_k \leq r(P)\}.$$

In the $k = 1$ case we also say briefly that (P, w) has the *Sperner property* or (P, w) *is Sperner*. Further, (P, w) has the *strong Sperner property* ((P, w) *is strongly Sperner*) if (P, w) has the k-Sperner property for all $k = 1, 2, \ldots$.

A sequence of nonnegative real numbers $\{a_n\}$ is called *unimodal* if there is a number h such that $a_i \leq a_{i+1}$ for $i < h$ and $a_i \geq a_{i+1}$ if $i \geq h$. It is called *logarithmically concave* (or *log concave*) if $a_i^2 \geq a_{i-1}a_{i+1}$ for all i. For a finite sequence $(a_0, \ldots, a_n)$, we say that it is *symmetric* if $a_i = a_{n-i}$ for all i. If the (weighted) Whitney numbers of (P, w) are unimodal (resp. symmetric), then (P, w) is said to be *rank unimodal* (resp. *rank symmetric*). If $\{a_n\}$ and $\{b_n\}$ are two infinite sequences of real numbers, the following notations for $n \to \infty$ are well known: $a_n \sim b_n$ if $\lim_{n\to\infty} a_n/b_n = 1$; $a_n = O(b_n)$ if there exists some $c \in \mathbb{R}$ such that $|a_n| \leq c|b_n|$ for all n; $a_n = o(b_n)$ if $\lim_{n\to\infty} a_n/b_n = 0$ and $a_n \lesssim b_n$ if $a_n \leq b_n(1 + o(1))$. All logarithms in this book are to the basis $e = 2.718\ldots$.

As usual, we denote the largest integer that is not greater than a given real number x by $\lfloor x \rfloor$. For the smallest integer that is not smaller than x, we write $\lceil x \rceil$. The set $\{1, \ldots, n\}$ we abbreviate by $[n]$. For the family of k-element subsets and for the power set of $[n]$, we use the notation $\binom{[n]}{k}$ (resp. $2^{[n]}$), which is motivated by the corresponding sizes. $A \subseteq B$ means that A is a subset of B, whereas strict inclusion is denoted by $A \subset B$. For the set difference of sets A and B, we write $A - B$. Moreover, we denote the complement of A in $[n]$ by $\overline{A}$; that is, $\overline{A} := [n] - A$. For $\mathcal{F} \subseteq 2^{[n]}$, let $\overline{\mathcal{F}} := \{\overline{A} : A \in \mathcal{F}\}$ be the *complementary family*.

Before considering some concrete examples of posets and lattices in the next section, let us look at some larger classes of ranked posets. For a general study of these lattices, see, for example, Aigner [21] and Stanley [441]. In a lattice P, the elements covering the minimal element are called *atoms*. The rank function of P is called *modular* (resp. *semimodular*) if

$$r(p \wedge q) + r(p \vee q) = (\text{ resp. } \leq) \, r(p) + r(q) \text{ for all } p, q \in P.$$

A (finite) lattice is called *modular* if it has a modular rank function. Moreover, a lattice is said to be *geometric* if it has a semimodular rank function and every

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element is a supremum of atoms. Finally a lattice is called *distributive* if the following identities hold for all $p, q, v \in P$:

$$p \wedge (q \vee v) = (p \wedge q) \vee (p \wedge v),$$

$$p \vee (q \wedge v) = (p \vee q) \wedge (p \vee v).$$

We note that each of these identities implies the other. All (finite) distributive lattices are ranked, for the proof see, for example, [21, p. 38]. Obviously, distributivity implies modularity.

1.3. The main examples

The following are several examples of posets we will consider in this book. Most of these posets can easily be shown to be lattices. Hence in all traditional examples the word *lattice* will be used instead of *poset*. Further, it is mentioned without proof that all following posets are ranked. The reader will learn in the book that all posets up to the last one have the Sperner property.

Example 1.3.1. *The Boolean lattice B_n.*

The poset of all subsets of an n-element set, ordered by inclusion, is the Boolean lattice. Obviously, B_n is isomorphic to $(0 < 1)^n$ as well as to the poset of all faces of an $(n-1)$-dimensional simplex (including the empty set as a face), ordered by inclusion. It is easy to see that $N_k(B_n)$ consists of all k-element subsets; thus $W_k(B_n) = \binom{n}{k}$. The Hasse diagrams of B_3 and B_4 are illustrated in Figure 1.1.

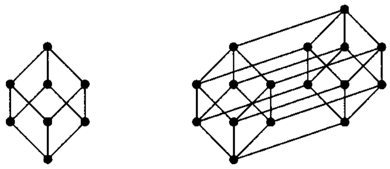

Figure 1.1

Example 1.3.2. *Chain products $S(k_1, \ldots, k_n)$.*

The poset $S(k_1, \ldots, k_n)$ consists of all n-tuples of integers $\boldsymbol{a} = (a_1, \ldots, a_n)$ such that $0 \le a_i \le k_i$, $i = 1, \ldots, n$, and we have $\boldsymbol{a} \le \boldsymbol{b}$ iff $a_i \le b_i$ for all i. We will adopt the convention $k_1 \ge \cdots \ge k_n$. Obviously, $S(k_1, \ldots, k_n) \cong \prod_{i=1}^{n}(0 < 1 < \cdots < k_i)$, and therefore $S(k_1, \ldots, k_n)$ is called a *chain product*. Given n distinct primes $p_1, \ldots, p_n$, $S(k_1, \ldots, k_n)$ is isomorphic to the *lattice*

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of all divisors of $p_1^{k_1} \dots p_n^{k_n}$, ordered by divisibility. The Boolean lattice B_n is isomorphic to $S(1, \dots, 1)$. From the product representation we obtain that the rank of an element $\boldsymbol{a}$ of $S(k_1, \dots, k_n)$ is given by $r(\boldsymbol{a}) = a_1 + \dots + a_n$, and the rank-generating function is $\prod_{i=1}^{n}(1 + x + \dots + x^{k_i})$.

Example 1.3.3. *The cubical poset* Q_n.

Q_n is the *poset of all faces of an n-dimensional cube* (not including the empty set as a face), ordered by inclusion. Here we consider only the discrete cube $\{\boldsymbol{a} = (a_1, \dots, a_n) : a_i \in \{0, 1\}\}$. Its faces are all subsets of the form $\{\boldsymbol{a} : a_i \in \{0, 1\}$ if $i \notin I, a_i = \alpha_i$ if $i \in I (i = 1, \dots, n)\}$, where I is a subset of $[n]$ and $\alpha_i (i \in I)$ are fixed elements of $\{0, 1\}$. Clearly the faces are exactly the intervals in the Boolean lattice B_n. If one notes that a face corresponds to an n-tuple $\boldsymbol{b} = (b_1, \dots, b_n)$ where $b_i = 2$ if $i \notin I$ and $b_i = \alpha_i$ if $i \in I$, then it is not difficult to see that Q_n is isomorphic to a product of n factors given in Figure 1.2. Thus we consider Q_n

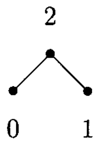

Figure 1.2

mostly as the set of all n-tuples $\boldsymbol{b} = (b_1, \dots, b_n)$ with $b_i \in \{0, 1, 2\}$ for all i and ordered by $\boldsymbol{b} \le \boldsymbol{c}$ iff $c_i = 2$ or $b_i = c_i$ for all i. Obviously, the rank of $\boldsymbol{b}$ equals the number of "twos" in $\boldsymbol{b}$ and $W_k(Q_n) = \binom{n}{k} 2^{n-k}$.

We get the *cubical lattice* $\hat{Q}_n$ if we add to Q_n a minimal element (which is smaller than all elements of Q_n).

Example 1.3.4. *The function poset* F_k^n.

F_k^n consists of all partially defined functions of an n-element set into a k-element set. For a function f of F_k^n, let $D(f)$ be its domain. Two elements of F_k^n are ordered in the following way: $f \le g$ iff $D(f) \subseteq D(g)$ and $f(x) = g(x)$ for all $x \in D(f)$. A partially defined function of $\{x_1, \dots, x_n\}$ into $\{y_1, \dots, y_k\}$ can be represented as an n-tuple $\boldsymbol{a} = (a_1, \dots, a_n)$, where $a_i = 0$ if $x_i \notin D(f)$ and $a_i = j$ if $x_i \in D(f)$ and $f(x_i) = y_j$. Thus F_k^n is isomorphic to a product of n factors given in Figure 1.3. It follows that F_2^n is isomorphic to the dual of Q_n. The rank of an element $\boldsymbol{a}$ of the poset F_k^n is given by the number of nonzero elements in $\boldsymbol{a}$, and we have $W_i(F_k^n) = \binom{n}{i} k^i$. If we add a maximal element to F_k^n, we get the *function lattice* $\hat{F}_k^n$.

Example 1.3.5. *Star products* $T(k_1, \dots, k_n)$.

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