ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Infinite Dimensional Optimization and Control Theory

H. O. FATTORINI

CAMBRIDGE
UNIVERSITY PRESS

PUBLISHED BY THE PRESS SYNDICATE OF THE UNIVERSITY OF CAMBRIDGE
The Pitt Building, Trumpington Street, Cambridge, United Kingdom

CAMBRIDGE UNIVERSITY PRESS
The Edinburgh Building, Cambridge CB2 2RU, UK http://www.cup.cam.ac.uk
40 West 20th Street, New York, NY 10011-4211, USA http://www.cup.org
10 Stamford Road, Oakleigh, Melbourne 3166, Australia

First published 1999

Printed in the United States of America

Typeset in 10/12 Times Roman in LaTeX 2_ε [TB]

A catalog record for this book is available from the British Library

Library of Congress Cataloging-in-Publication Data

Fattorini, H. O. (Hector O.), 1938–
Infinite dimensional optimization and control theory / H. O. Fattorini.
p. cm. – (Cambridge studies in advanced mathematics: 54)
ISBN (invalid) 0-521-42125-6 (ho)
1. Mathematical optimization. 2. Calculus of variations.
3. Control theory. I. Title. II. Series.
QA402.5.F365 1996
003′.5 – dc20 95-47377
CIP

ISBN 0 521 45125 6 hardback

CONTENTS

1

Calculus of Variations and Control Theory

1.1. Calculus of Variations: Surface of Revolution of Minimum Area

Consider the surfaces Σ of revolution about the x-axis whose boundary consists of the circles

$$\Gamma_a = \{(x, y, z); x = a, y^2 + z^2 = r_a\}$$
$$\Gamma_b = \{(x, y, z); x = b, y^2 + z^2 = r_b\}$$

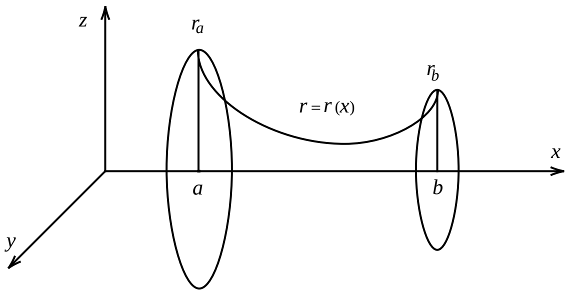

Figure 1.1.

$(a < b, r_a, r_b \geq 0)$. Among these surfaces, we look for one having minimum area. If $r = r(x) \geq 0$ $(a \leq x \leq b)$ is the equation of the surface, the area is[(1)]

$$A(r) = 2\pi \int_a^b r(x)\sqrt{1 + r'(x)^2}dx.$$

(1) If $r(x)$ is negative for some x the expression for the area is incorrect unless one replaces $r(x)$ by $|r(x)|$ in the integrand. This is ignored in some textbook treatments.

Since Γ_a and Γ_b are the boundary of Σ,

$$r(a) = r_a, \qquad r(b) = r_b. \tag{1.1.1}$$

More generally, we may minimize

$$J(y) = \int_a^b F(x, y(x), y'(x))dx. \tag{1.1.2}$$

The "variable" in J is itself a function $y(x)$ defined in the interval $a \le x \le b$. An expression like this is called a **functional**; the **admissible** functions $y(x)$ in the integrand belong to the space $C^{(1)}[a, b]$ of continuously differentiable functions in $a \le x \le b$ and satisfy boundary conditions, in this case (1.1.1). Admissible functions are an *affine* subspace of $C^{(1)}[a, b]$: if $y(x)$ satisfies (1.1.1) and $v(x)$ belongs to the subspace $C_0^{(1)}[a, b]$ of $C^{(1)}[a, b]$ defined by $v(a) = v(b) = 0$ then

$$y(x) + hv(x) \tag{1.1.3}$$

is admissible for any h. This makes viable the argument below, where we assume that $F(x, y, y')$ is everywhere defined and continuously differentiable with respect to y and y', with F, $\partial F/\partial y$, $\partial F/\partial y'$ continuous in all variables.

Assume that $\bar{y}(\cdot) \in C^{(1)}[a, b]$ is a **minimizing element** or a **minimum** of $J(y)$ (that is, $J(\bar{y}) \le J(y)$ for all admissible $y(\cdot)$). Let $v(\cdot) \in C_0^{(1)}[a, b]$. Then

$$J(\bar{y} + hv) \ge J(\bar{y})$$

for all real h, hence $\phi(h) = J(\bar{y} + hv)$ has a minimum at $h = 0$. This implies $\phi'(0) = 0$. This condition can be written

$$\begin{aligned}
\phi'(0) &= \frac{d}{dh}\bigg|_{h=0} \phi(h) = \frac{d}{dh}\bigg|_{h=0} J(\bar{y} + hv) \\
&= \frac{d}{dh}\bigg|_{h=0} \int_a^b F(x, \bar{y}(x) + hv(x), \bar{y}'(x) + hv'(x))dx \\
&= \int_a^b \left\{ \frac{\partial F}{\partial y}(x, \bar{y}(x), \bar{y}'(x))v(x)dx + \frac{\partial F}{\partial y'}(x, \bar{y}(x), \bar{y}'(x))v'(x) \right\} dx = 0
\end{aligned} \tag{1.1.4}$$

for any $v(\cdot) \in C_0^{(1)}[a, b]$.

Lemma 1.1.1. *Let $f(x), g(x)$ be continuous in $a \le x \le b$. Assume that for every $v \in C_0^{(1)}[a, b]$ we have*

$$\int_a^b \{f(x)v(x) + g(x)v'(x)\}dx = 0. \tag{1.1.5}$$

Then (after possible modification in a null set) $g(\cdot) \in C^{(1)}[a, b]$ and $g'(x) \equiv f(x)$.

Proof. Assume first that $f(x) \equiv 0$; (1.1.5) and the boundary conditions imply

$$\int_a^b (g(x) - c)v'(x)dx = 0 \tag{1.1.6}$$

for any c. Define

$$v(x) = \int_a^x (g(\xi) - c_0)d\xi$$

where c_0 is such that $v(b) = 0$; then $v(\cdot) \in C_0^{(1)}[a, b]$. Replacing this particular $v(\cdot)$ and c_0 in (1.1.6) we obtain $g(x) \equiv c_0$. In the general case, define

$$F(x) = \int_0^x f(\xi)\, d\xi$$

and integrate (1.1.5) by parts, obtaining

$$\int_a^b \{g(x) - F(x)\}v'(x)dx = 0,$$

for all $v(\cdot) \in C_0^{(1)}[a, b]$ so that $g(x)$ and $F(x)$ differ by a constant. ■

Using this result in (1.1.4) we deduce that $\partial F(x, \overline{y}(x), \overline{y}'(x))/\partial y'$ is a continuously differentiable function of x and that $y(x) = \overline{y}(x)$ satisfies the **Euler equation**

$$\frac{\partial F}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx}\frac{\partial F}{\partial y'}(x, y(x), y'(x)) = 0. \tag{1.1.7}$$

Hence (assuming J has a minimum in $C^{(1)}[a, b]$), the minimization problem reduces to solving (1.1.7) with boundary conditions (1.1.1). However, the theory of boundary value problems for differential equations is not as simple as that of initial value problems (where the value of a solution and its derivative are given at a single point). For a glimpse on boundary value problems see Elsgolts [1970, p. 165] or Gelfand–Fomin [1963, p. 16]; for a more complete treatment, see Keller [1968].

If $y(\cdot)$ is twice continuously differentiable we may apply the chain rule in the right side of (1.1.7) and obtain a *bona fide* second order differential equation for $\overline{y}(\cdot)$ (however, $y(x)$ may not be so smooth; see Example 1.1.3). If $y(\cdot) \in C^{(2)}[a, b]$ and, in addition, $F(x, y, y') = F(y, y')$ is independent of x, we multiply (1.1.7) by $y'(x)$ and integrate, obtaining

$$F(y(x), y'(x)) - y'(x)\frac{\partial F}{\partial y'}(y(x), y'(x)) = \beta, \tag{1.1.8}$$

where β is a constant. Conversely, any solution of (1.1.8) with $y'(x) \neq 0$ necessarily satisfies the Euler equation.

The Euler equation for the minimal area problem is

$$\sqrt{1+r'^2} - \frac{d}{dx}\frac{rr'}{\sqrt{1+r'^2}} = 0.^{(2)} \tag{1.1.9}$$

Equation (1.1.8) is

$$r(x) = \beta\sqrt{1+r'(x)^2} \tag{1.1.10}$$

with solution $r(x) \equiv 0$ for $\beta = 0$, and

$$r(x) = \beta\cosh\left(\frac{x-\alpha}{\beta}\right) \tag{1.1.11}$$

for $\beta \neq 0$, where α is arbitrary. (Gelfand–Fomin [1963, p. 20]).

Example 1.1.2. There exist either two, one, or no $r(x)$ of the form (1.1.11) satisfying the boundary conditions (1.1.1). For a proof, see Bliss [1925, p. 90] or Cesari [1983, p. 143]. We just take a look at the case $a = 0$, $b = 1$, $r_a = r_b = r$. To satisfy the boundary conditions we must make $\alpha = 1/2$ and solve the transcendental equation

$$\phi(\beta) = \beta\cosh\left(\frac{1}{2\beta}\right) = r. \tag{1.1.12}$$

If m is the minimum of the function of $\beta > 0$ on the left-hand side, then the equation has no solution if $r < m$, one solution if $r = m$, and two solutions if $r > m$. We have $m = 0.7544\ldots$, attained at $\beta = 0.4167\ldots$.

For $r_a = r_b = 1$ (1.1.12) has two solutions, $\beta = 0.2350\ldots$ and $\beta = 0.8483\ldots$.

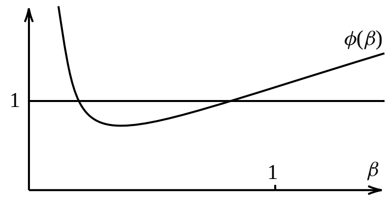

Figure 1.2.

Using formula (1.1.11) the area integral reduces to

$$2\pi\beta\int_0^1 \cosh^2((2x-1)/2\beta)dx = \pi\beta + \pi\beta^2\sinh(1/\beta).$$

The surface corresponding to $\beta = 0.2350\ldots$ (resp. to $\beta = 0.8483\ldots$) has area $6.8456\ldots$ (resp. $5.9918\ldots$).

(2) A minimizing element $\bar{r}(x)$ of $A(r)$ satisfies the Euler equation (1.1.9) if $\bar{r}(x) > 0$; otherwise, $\bar{r}(x) + hv(x)$ may be zero for h arbitrarily small invalidating (1.1.4). Note also that no solution of (1.1.9) may be zero anywhere.

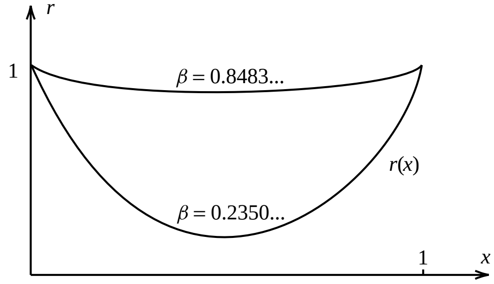

Figure 1.3.

Obviously, the first surface cannot be a minimum. The second is, although this is far from obvious; in fact, it is not even clear whether a minimal surface exists. On the other hand, if $r_a = r_b = m$, the only solution of the form (1.1.11) satisfying the boundary conditions (1.1.1), whose area is $4.2903\ldots$ is not a minimum of the functional. In fact, the "surface" consisting of the two disks spanned by Γ_0 and Γ_1 connected by the segment of the x-axis between them has area $2\pi\beta^2\cosh^2(1/2\beta) = 3.5762\ldots < 4.2903\ldots$. Obviously, this is not one of the surfaces allowed to compete for the minimum, but it can be approximated by smooth surfaces of revolution having almost the same area, for instance, $r_n(x) = (x^n + (1-x)^n)\beta\cosh(1/2\beta)$ for large n (see Figure 1.4).

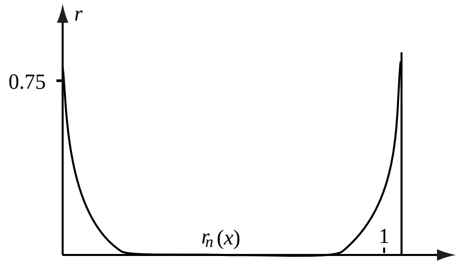

Figure 1.4.

Taking $r_a = r_b = m' > m$ with m' sufficiently close to m we obtain two functions of the form (1.1.11) satisfying the boundary conditions, none of which is a minimum.

For a complete solution of the minimal surface problem in the spirit of control theory, see **10.5**; classical treatments are given in Bliss [1925, Ch. IV] or Cesari [1983, p.143].

The results on the functional (1.1.2) extend easily to functionals depending on n functions $y_1(x), \ldots, y_n(x)$ and their derivatives,

$$J(y_1, \ldots, y_n) = \int_a^b F(x, y_1(x), \ldots, y_n(x), y_1'(x), \ldots, y_n'(x))dx. \qquad (1.1.13)$$

Dealing with each $y_j(x)$ separately, we deduce that if F is everywhere defined and continuously differentiable with respect to each y_j and y_j' with F and partial derivatives continuous in all arguments, then a minimum $\overline{y}_1(x), \ldots, \overline{y}_n(x)$ of (1.1.11) where each $\overline{y}_j(x)$ belongs to $C^{(1)}[a, b]$ must satisfy the **Euler equations**

$$\frac{\partial F}{\partial y_j} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_j'}\right) = 0 \quad (j = 1, 2, \ldots, n). \tag{1.1.14}$$

Example 1.1.3. (Gelfand–Fomin [1963, p. 16]) The functional

$$J(y) = \int_{-1}^{1} y^2(x)\big(2x - y'(x)\big)^2 dx$$

with boundary conditions $y(-1) = 0, y(1) = 1$ attains its minimum (zero) for $\overline{y}(x) = 0 \ \ (x \le 0), \overline{y}(x) = x^2 \ \ (x \ge 0)$. The minimum $\overline{y}(\cdot)$ does not belong to $C^{(2)}[a, b]$.

Example 1.1.4. (Gelfand–Fomin [1963, p. 17]) Assume $F(x, y, y')$ has continuous partials up to order two in all variables. Let $\overline{y}(\cdot) \in C^{(1)}[a, b]$ be a solution of Euler's equation (1.7) with

$$\frac{\partial^2 F}{\partial y'^2}(x, \overline{y}(x), \overline{y}'(x)) \neq 0 \quad (a \le x \le b). \tag{1.1.15}$$

Then $\overline{y}(\cdot) \in C^{(2)}[a, b]$. This applies to the minimal area problem (where $\partial^2 F(r, r')/\partial r'^2 = r(1 + r'^2)^{-3/2}$) as follows: if $r = r(x)$ is the equation of a minimal surface in $C^{(1)}[a, b]$ with $r(x) > 0$, then $r(\cdot) \in C^{(2)}[a, b]$.

1.2. Interpretation of the Results

All we have shown on the problem of minimizing (1.1.2) is that if $\overline{y}(\cdot) \in C^{(1)}[a, b]$ is a minimum, then $\overline{y}(\cdot)$ satisfies the Euler equation (1.1.7). Thus, we only have *necessary* conditions for a minimum. They may not be sufficient: a solution of (1.1.7) satisfying the boundary conditions may not be a minimum of $J(y)$, as we have seen in Example 1.1.2. We meet the same problem in calculus trying to find the minima of a function $f(x) = f(x_1, x_2, \ldots, x_m)$ in m-dimensional Euclidean space $\mathbb{R}^m$: at a minimum $\overline{x} = (\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_m)$ of f we have

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_m} = 0 \tag{1.2.1}$$

but these conditions are not sufficient. Points $\overline{x} \in \mathbb{R}^m$ where (1.2.1) holds are called **extremals** of the function f, and we use the same terminology for the functional (1.1.2): a function $y \in C^{(1)}[a, b]$ satisfying (1.1.7) and the boundary conditions is an **extremal** of J. For instance, the inner surface in Figure 1.3 for $r_a = r_b = 1$ is an extremal but not a minimum.

In some cases, necessary conditions in combination with existence theorems give the actual minima of a functional. For instance, if a minimizing element $\bar{y}(\cdot) \in C^1[a, b]$ exists and solutions of the boundary value problem are unique, then the solution of the boundary value problem must be the minimum. However, this may fail as seen in Example 1.1.2 for the minimal area problem; solutions in $C^{(1)}[a, b]$ may not exist or the boundary value problem may have multiple solutions. Another problem without smooth solutions is

Example 1.2.1. (Gelfand–Fomin [1963, p. 61]) The functional

$$J(y) = \int_{-1}^{1} y^2(x)(1 - y'(x))^2 dx$$

with boundary conditions $y(-1) = 0, y(1) = 1$ attains its minimum (zero) for $\bar{y}(x) = 0 \;\; (x \le 0), \bar{y}(x) = x \;\; (x \ge 0)$. The minimum $\bar{y}(\cdot)$ does not belong to $C^{(1)}[a, b]$.

Proper treatment of variational problems (and of control problems) needs a less demanding definition of solution; see **10.5** for more on this.

1.3. Mechanics and Calculus of Variations

Consider a mechanical system with a finite number of degrees of freedom. We denote by $q_1, q_2, \ldots, q_n$ the **generalized coordinates** of the system, in terms of which the Cartesian coordinates can be determined in 3-space. The n-dimensional point $q = (q_1, q_2, \ldots, q_n)$ moves arbitrarily in a region of Euclidean space $\mathbb{R}^n$ or, more generally, in an n-dimensional differential manifold. Assume for simplicity that the system consists of a finite number of particles with Cartesian coordinates $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p \in \mathbb{R}^3$:

$$\mathbf{r}_j = \mathbf{r}_j(q_1, \ldots, q_n) \quad (1 \le j \le p), \tag{1.3.1}$$

and that the forces acting on the system are due to a potential,

$$F_j = -\nabla_{\mathbf{r}_j} U(\mathbf{r}_1, \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_p) \quad (j = 1, 2, \ldots, p).$$

The Lagrangian of the system is

$$L = \sum_{j=1}^{p} \frac{m_j}{2} \|\mathbf{r}'_j\|^2 - U(\mathbf{r}_1, \ldots, \mathbf{r}_p) = T - U \tag{1.3.2}$$

where the **kinetic energy** T and the **potential energy** U are expressed in terms of the generalized coordinates. The motion of the system is described by **Hamilton's**

principle (Kompaneyets [1978, p. 17]): the possible motions $q_1(t), q_2(t), \ldots, q_n(t)$ of the system in a time interval $t_0 \le t \le t_1$ are extremals of the **action integral**

$$S = \int_{t_0}^{t_1} L(q_1, q_2, \ldots, q_n, q_1', q_2', \ldots, q_n')dt, \tag{1.3.3}$$

that is, they solve the Euler equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_j'}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{1.3.4}$$

((see (1.1.14)). These are the **Euler-Lagrange equations** of mechanics and are combined with initial or boundary conditions: usually, the initial position and velocity of the system (that is, the position and velocity at $t = t_0$) are given. It also makes sense to specify the position of the system at different times t_0 and t_1, which produces a boundary value problem in the sense of **1.1.**

Example 1.3.1. A **simple pendulum** is a particle of mass m connected to the origin by a rigid rod of length l and allowed to move on the (x, y)-plane.

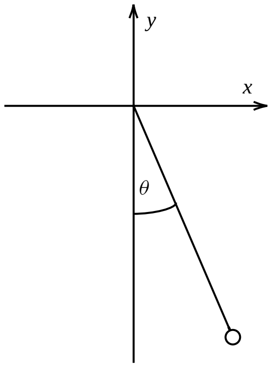

Figure 1.5.

Only one generalized coordinate, the angle θ, is necessary. It moves in the circle obtained identifying points modulo 2π on the line. The force of gravity comes from the potential energy $U = mgy$ (g the acceleration of gravity). Since $\mathbf{r} = l(\sin\theta, -\cos\theta)$, $\mathbf{r}' = l\theta'(\cos\theta, \sin\theta)$ and

$$L = T - U = \frac{m}{2}l^2\theta'^2 - mgl(1 - \cos\theta), \tag{1.3.5}$$

(where we have taken arbitrarily the stable equilibrium position as having potential energy zero). The Euler-Lagrange equation is the nonlinear pendulum equation

$$\theta'' + \frac{g}{l}\sin\theta = 0. \tag{1.3.6}$$

1.4. Optimal Control: Fuel Optimal Landing of a Space Vehicle

A space vehicle on a vertical trajectory tries to land smoothly (that is, with velocity zero) on the surface of a planet (see Figure 1.6). Denote by $h(t)$ the height at time t (so that $v(t) = h'(t)$ is the instantaneous velocity). Since combustible is being consumed, the mass $m(t)$ of the vehicle is a nonincreasing function of t. If we call $u(t)$ the instantaneous upwards thrust, Newton's law gives $m(t)h''(t) = -gm(t) + u(t)$, where g is the acceleration of gravity. Assuming that the thrust is proportional to the rate of decrease of mass (that is, proportional to the rate at which combustible is used up) we introduce $v(t) = h'(t)$ as a variable and obtain the following first-order system of differential equations:

$$h'(t) = v(t), \qquad v'(t) = -g + \frac{u(t)}{m(t)}, \qquad m'(t) = -Ku(t),$$

where $K > 0$. At the initial time $t_0 = 0$ we have initial conditions

$$h(0) = h_0, \qquad v(0) = v_0, \qquad m(0) = m_0.$$

The vehicle will land softly at time $\bar{t} \geq 0$ if

$$h(\bar{t}) = 0, \qquad v(\bar{t}) = 0.$$

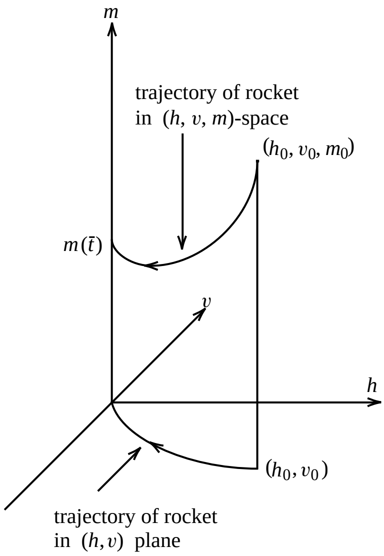

Figure 1.6.

The thrust cannot be negative or arbitrarily large:

$$0 \leq u(t) \leq R$$

for some $R > 0$. We have an optimization problem if we try to land minimizing the amount of combustible

$$m(0) - m(\bar{t}) = K \int_0^{\bar{t}} u(\tau) d\tau = J(u)$$

consumed from $t = 0$ until the landing time $t = \bar{t}$.

A complete solution of the landing problem is given in **4.7** and **4.8**.

1.5. Optimal Control Problems Described by Ordinary Differential Equations

The rocket landing problem is a particular case of a general **optimal control problem**: minimize a **cost functional** or **performance index** of the form

$$y_0(\bar{t}, u) = \int_0^{\bar{t}} f_0(\tau, y(\tau), u(\tau)) d\tau + g_0(\bar{t}, y(\bar{t})) \tag{1.5.1}$$

among all the solutions (or **trajectories**) of the vector differential equation

$$y'(t) = f(t, y(t), u(t)), \qquad y(0) = \zeta \tag{1.5.2}$$

with $y(t)$ and $f(t, y, u)$ m-vector functions and ζ a m-vector. The system (1.5.2) is called the **state equation** of the system. The **control** $u(t)$ is a k-vector function satisfying a **control constraint**

$$u(t) \in U \tag{1.5.3}$$

where $U \subseteq \mathbb{R}^k$ is the **control set** (more generally, we may use control sets $U(t)$ depending on time). Controls satisfying (1.5.3) are called **admissible**. In general, the problem includes a **target condition**

$$y(\bar{t}) \in Y \tag{1.5.4}$$

where $Y \subseteq \mathbb{R}^m$ is the **target set**. The **terminal time** $\bar{t}$ at which the target condition (1.5.4) is to be satisfied may be fixed or free. The problem may also include **state constraints**

$$y(t) \in M(t) \tag{1.5.5}$$

to be satisfied in $0 \le t \le \bar{t}$.

To fit the rocket landing problem in this scheme, we take $m = 3, k = 1$, $y(t) = (h(t), v(t), m(t))$, $f(t, y, u) = (v, -g + u/m, -Ku)$, $\zeta = (h_0, v_0, m_0)$, and $U = [0, R]$. The target set is the half line $Y = \{0\} \times \{0\} \times [m_e, \infty)$, and in the cost functional, we have $f_0(t, y, u) = Ku, g_0 = 0$. The problem actually includes two state constraints, namely

$$h(t) \ge 0, \qquad m(t) \ge m_e > 0. \tag{1.5.6}$$

The second says that the mass at time t cannot be less than the mass m_e of the rocket with empty fuel tanks, and is automatically satisfied since $u(t) \geq 0$; the first warns that we must not drive the rocket into the ground. These state constraints are of the form (1.5.5) with $M(t) = [0, \infty) \times (-\infty, \infty) \times [m_e, \infty)$ for all t.

A prerequisite to the solution of the general optimal control problem is the **controllability problem**: can we find an admissible control $u(t)$ such that the corresponding trajectory $y(t)$ with $y(0) = \zeta$ satisfies the target condition (1.5.4) and the state constraint (1.5.5)? This controllability problem may not have a solution. For instance, in the landing problem, Y cannot be hit at all (that is, soft landing is impossible) if the initial amount of combustible m_0 is insufficient.

1.6. Calculus of Variations and Optimal Control. Spike Perturbations

Optimal control problems are similar to problems of calculus of variations; both deal with minimizing functionals. One may try to apply to control problems the arguments in **1.1**, based on affine perturbations $\bar{u}(t) + hv(t)$ of an optimal control $\bar{u}(t)$. However, it is not clear how to take h and $v(t)$ in order that the target condition (1.5.4) be satisfied by the trajectory corresponding to $u(t) + hv(t)$. Even if we ignore the target condition, we must be sure that the control $\bar{u}(t) + hv(t)$ is admissible, that is,

$$\bar{u}(t) + hv(t) \in U.$$

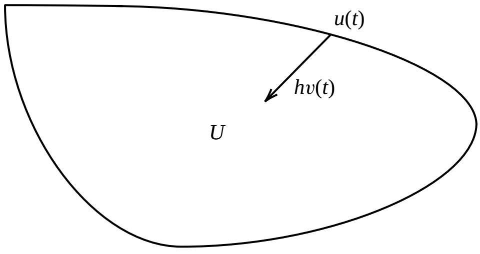

Figure 1.7.

For $h > 0$, this requires $v(t)$ to "point into U" at $\bar{u}(t)$ (see Figure 1.7). **Spike** or **needle** perturbations are better suited to control constraints and are defined as follows. Let $\bar{t} > 0$, $0 < s \leq \bar{t}$, $0 \leq h \leq s$ and v an element of the control set U. Given an admissible control $u(t)$ we define a new control $u_{s,h,v}(t)$ by

$$u_{s,h,v}(t) = \begin{cases} v & (s - h < t \leq s) \\ u(t) & \text{elsewhere} \end{cases} \tag{1.6.1}$$

(see Figure 1.8).

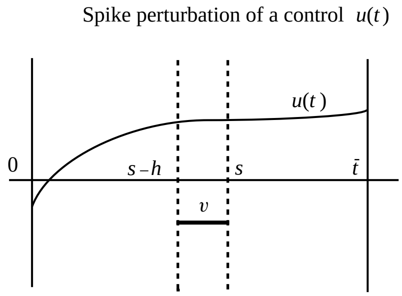

Figure 1.8.

Obviously, $u_{s,h,v}(t)$ is an admissible control. If $\bar{u}(t)$ is optimal in an interval $0 \le t \le \bar{t}$,

$$y_0(\bar{t}, \bar{u}_{s,h,v}) \ge y_0(\bar{t}, \bar{u}) \tag{1.6.2}$$

for s, h, v arbitrary, thus if

$$\xi_0(t) = \frac{d}{dh}\bigg|_{h=0+} y_0(t, \bar{u}_{s,h,v}) = \lim_{h\to 0+} \frac{1}{h}\big(y_0(t, \bar{u}_{s,h,v}) - y_0(t, \bar{u})\big) \tag{1.6.3}$$

exists, we have

$$\xi_0(\bar{t}) \ge 0 \tag{1.6.4}$$

for s, v arbitrary.

Spike perturbations can also be defined for $h < 0$ (the spike stands to the right of s). Since $y_0(\bar{t}, \bar{u}_{s,h,v}) \ge y_0(\bar{t}, \bar{u})$ for h of any sign, this should improve (1.6.4) to $\xi_0(\bar{t}) = 0$ for all s, v. However, the function $h \to y_0(\bar{t}, \bar{u}_{s,h,v})$ may not have a two-sided derivative at $h = 0$ (see Example 1.6.1).

We compute *formally* $\xi_0(t)$ for arbitrary t; justification is postponed to **2.3**. The first step is to calculate

$$\xi(t) = \frac{d}{dh}\bigg|_{h=0+} y(t, u_{s,h,v}) = \lim_{h\to 0+} \frac{1}{h}\big(y(t, u_{s,h,v}) - y(t, u)\big), \tag{1.6.5}$$

where $y(t, u)$ denotes the solution of (1.5.2) corresponding to the control $u(t)$. We have

$$\begin{aligned} \xi'(t) &= \lim_{h\to 0+} \frac{1}{h}\big(y'(t, \bar{u}_{s,h,v}) - \bar{y}'(t, \bar{u})\big) \\ &= \lim_{h\to 0+} \frac{1}{h}\big\{ f(t, y(t, \bar{u}_{s,h,v}), \bar{u}_{s,h,v}(t)) - f(t, y(t, \bar{u}_{s,h,v}), \bar{u}(t))\big\} \\ &\quad + \lim_{h\to 0+} \frac{1}{h}\big\{ f(t, y(t, \bar{u}_{s,h,v}), \bar{u}(t)) - f(t, y(t, \bar{u}), \bar{u}(t))\big\}. \end{aligned} \tag{1.6.6}$$

The function $f(t, y(t, \bar{u}_{s,h,v}), \bar{u}_{s,h,v}(t)) - f(t, y(t, \bar{u}_{s,h,v}), \bar{u}(t))$ is zero except in $s - h < t \le s$, where $\bar{u}_{s,h,v}(t) = v$. Assuming that $y(t, \bar{u}_{s,h,v}) \approx y(s, \bar{u})$ and $\bar{u}(t) \approx \bar{u}(s)$ in the interval $s - h \le t \le s$ for h small enough, the first limit on the right should be the same as

$$\lim_{h\to 0+} \frac{1}{h} \chi_h(t)\{ f(s, y(s, \bar{u}), v) - f(s, y(s, \bar{u}), \bar{u}(s))\}$$

($\chi_h(t)$ the characteristic function of $s - h < t \le s$), which equals

$$\{ f(s, y(s, \bar{u}), v) - f(s, y(s, \bar{u}), u(s))\}\delta(t - s)$$

(δ the Dirac delta). The limit in the second term is computed by the chain rule. We obtain in this way the **variational equation** for $\xi(t)$,

$$\xi'(t) = \partial_y f(t, y(t, \bar{u}), \bar{u}(t))\xi(t) + \{ f(s, y(s, \bar{u}), v) - f(s, y(s, \bar{u}), \bar{u}(s))\}\delta(t - s) \\ (0 \le t \le \bar{t}), \quad \xi(0) = 0. \quad (1.6.7)$$

Equivalently, $\xi(t) = 0$ for $t < s$ and

$$\begin{aligned} \xi'(t) &= \partial_y f(t, y(t, \bar{u}), \bar{u}(t))\xi(t) \quad (s \le t \le \bar{t}), \\ \xi(s) &= \{ f(s, y(s, \bar{u}), v) - f(s, y(s, \bar{u}), \bar{u}(s))\}. \end{aligned} \quad (1.6.8)$$

In both equations, $\partial_y f$ denotes the Jacobian matrix of f with respect to the y variables. To figure out the limit (1.6.3) for a cost functional of the form (1.5.1), we write the integrand in the form

$$\begin{aligned} &f_0(\tau, y(\tau, \bar{u}_{s,h,v}), \bar{u}_{s,h,v}(\tau)) - f_0(\tau, y(\tau, \bar{u}), \bar{u}(\tau)) \\ &= \{ f_0(\tau, y(\tau, \bar{u}_{s,h,v}), \bar{u}_{s,h,v}(\tau)) - f_0(\tau, y(\tau, \bar{u}_{s,h,v}), \bar{u}(\tau))\} \\ &\quad + \{ f_0(\tau, y(\tau, \bar{u}_{s,h,v}), \bar{u}(\tau)) - f_0(\tau, y(\tau, \bar{u}), \bar{u}(\tau))\} \end{aligned}$$

and argue as in the computation of (1.6.5). The final result is

$$\begin{aligned} \xi_0(t) &= \{ f_0(s, y(s, \bar{u}), v) - f_0(s, y(s, \bar{u}), \bar{u}(s))\} \\ &\quad + \int_s^t \langle \nabla_y f_0(\tau, y(\tau, \bar{u}), \bar{u}(\tau)), \xi(\tau)\rangle d\tau + \langle \nabla_y g_0(t, y(t, \bar{u})), \xi(t)\rangle. \end{aligned} \quad (1.6.9)$$

where ∇ denotes gradient and $\langle \cdot, \cdot \rangle$ inner product in $\mathbb{R}^m$. From all the transgressions in the argument, the worst is perhaps the continuity assumption that $\bar{u}(t) \approx \bar{u}(s)$ near s; optimal controls are often discontinuous. A correct proof needs some measure theory.

Replacing in (1.6.4), a necessary condition for $\bar{u}(\cdot)$ to be optimal is obtained. Some work will be needed in Chapter 2 to put this result in a usable form; we only show here how it works in a particular problem.

Example 1.6.1. Consider the control system

$$y'(t) = -u(t), \qquad y(0) = 1 \tag{1.6.10}$$

in the interval $0 \le t \le \bar{t}$, with control constraint

$$0 \le u(t) \le 1 \tag{1.6.11}$$

and cost functional

$$y_0(\bar{t}, u) = \int_0^{\bar{t}} u(\tau)^2 d\tau + y(\bar{t})^2. \tag{1.6.12}$$

Applications buffs may imagine a reservoir being pumped out at rate $u(t)$. The first term in the functional reflects the cost of pumping, thus minimization of $y_0(t, u)$ means draining the reservoir as much as possible while minimizing cost.

We have $f(t, y, u) \equiv -u$, so that the variational equation is

$$\xi'(t) = -\delta(t - s)\{v - u(s)\}, \qquad \xi(0) = 0$$

with solution $\xi(t) = -\nu(t - s)\{v - u(s)\}$, $\nu(\cdot)$ the Heaviside function $\nu(t) = 1(t \ge 0), u(t) = 0\ (t < 0)$. Assuming an optimal control $\bar{u}(t)$ exists and taking into account that $f_0(t, y, u) = u^2$, $g(t, y) = y^2$, (1.6.4) and (1.6.9) give $\xi_0(\bar{t}) = \{v^2 - \bar{u}(s)^2\} - 2y(\bar{t}, \bar{u})\{v - \bar{u}(s)\} \ge 0$, or $v^2 - 2y(\bar{t}, \bar{u})v \ge \bar{u}(s)^2 - 2y(\bar{t}, \bar{u})\bar{u}(s)$ for $0 \le s \le \bar{t}$ and $0 \le v \le 1$, so that

$$\bar{u}(s)^2 - 2y(\bar{t}, \bar{u})\bar{u}(s) = \min_{0 \le v \le 1}\{v^2 - 2y(\bar{t}, \bar{u})v\}. \tag{1.6.13}$$

This is a protoexample of Pontryagin's maximum (minimum) principle and shows one of its features: it gives the optimal control $\bar{u}(s)$, but only in terms of the unknown optimal trajectory $y(\bar{t}, \bar{u})$. In some cases (here for instance), it is possible to compute $\bar{u}(s)$ anyway. In fact, the initial value problem (1.6.10) and the fact that $0 \le \bar{u}(t) \le 1$ imply that $0 \le y(\bar{t}, \bar{u}) \le 1$. Accordingly, the minimum of $v^2 - 2y(\bar{t}, \bar{u})v$ in $0 \le v \le 1$ is $\bar{u}(s) \equiv y(\bar{t}, \bar{u})$. Replacing in the equation and making $t = \bar{t}$, we get $y(\bar{t}, \bar{u}) = 1 - \bar{t}y(\bar{t}, \bar{u})$, so that

$$\bar{u}(s) \equiv \frac{1}{1 + \bar{t}} \quad (0 \le s \le \bar{t}).$$

Note that if h is of arbitrary sign and $t > s + h$ we have

$$y(t, \bar{u}_{s,h,v}) = (t - |h|)\bar{u}^2 + |h|v^2 + (1 - (t - |h|)\bar{u} - |h|v)^2$$

for $t > s$; thus $h \to y(t, \bar{u}_{s,h,v})$ is not (two-sided) differentiable at $h = 0$. Computing the limit (1.6.3) for $h < 0$ just produces (1.6.4) again.

1.7. Optimal Control: Minimum Drag Nose Shape in Hypersonic Flow

A vehicle moves through a fluid in the direction of the positive y-axis with uniform speed V. Its nose is a body of revolution whose projection on the x, y plane is a curve described by parametric equations $x = x(t), y = y(t)$ $(0 \le t \le T)$ with $(x(0), y(0)) = (0, h)$, $(y(0), y(T)) = (r, 0)$; h is the **height** of the nose and r its **maximum radius**. The quotient r/h is the **fineness ratio**.

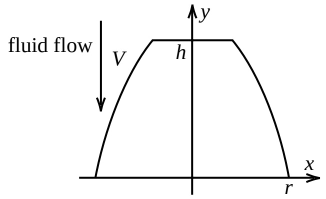

Figure 1.9.

A model due to Newton (see Goursat [1942, p. 658], McShane [1978/1989]) proposes that the drag normal to each surface element is proportional to the square $V^2 x'(t)^2/(x'(t)^2 + y'(t)^2)$ of the normal component of the velocity vector. The resultant of all these forces (obviously in the y-direction) is then the integral of $V^2 x'(t)^3/(x'(t)^2 + y'(t)^2)^{3/2}$ over the surface with respect to the area element $2\pi x(t)(x'(t)^2 + y'(t)^2)^{1/2}\,dt$, thus is proportional to the integral

$$\int_0^T \frac{x(t)x'(t)^3}{x'(t)^2 + y'(t)^2}\,dt.^{(1)} \qquad (1.7.1)$$

It is easy to see that, without further conditions, the minimum of the integral is $-\infty$, but there are physical reasons to consider only nondecreasing $x(t)$ and nonincreasing $y(t)$; if this is not the case (Figure 1.10) there may be parts of the surface of the body isolated from the flow by stagnant fluid or by other parts of the body.[(2)]

This problem is treated in textbooks such as Goursat [1942] using classical calculus of variations, but it admits a more natural formulation as a control problem. We set

$$x'(t) = u(t), \qquad y'(t) = -v(t), \qquad x(0) = 0, \qquad y(0) = h \qquad (1.7.2)$$

in a variable interval $0 \le t \le \bar{t}$ with control set $U = [0, \infty) \times [0, \infty)$, target condition

$$(x(\bar{t}), y(\bar{t})) = (r, 0), \qquad (1.7.3)$$

(1) For air flow, this model is said to be "…very good at hypersonic speeds but *not* very good at subsonic speeds" in Bryson–Ho [1969, p. 52]. Hence the title of this section.

(2) Ignoring these conditions has led some authors to brand Newton's drag model as "absurd." See McShane [1978/1989] for a refutation; careful reading of Newton's original formulation of the problem reveals that monotonicity of $x(t)$ and $y(t)$ is actually required. See also Goldstine [1980, p. 7].

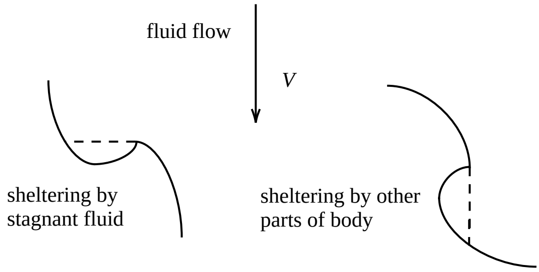

Figure 1.10.

and cost functional

$$y_0(t, u, v) = \int_0^t \frac{x(\tau)u(\tau)^3}{u(\tau)^2 + v(\tau)^2}\, d\tau. \tag{1.7.4}$$

A difference with the soft landing problem is that the control set is unbounded; thus, conditions on the controls $u(\cdot)$, $v(\cdot)$ are needed in order that (1.7.4) be finite. Since $xu^3/(u^2 + v^2) \le xu$, it is enough to require the controls to be integrable. Another difference is that the parameter t in the minimum drag nose shape problem has no physical meaning so that we are free to reparametrize the curve at our pleasure. This will be used in the solution of this problem, presented in **4.9**, **13.7** and **13.8**.

1.8. Control of Functional Differential Equations: Optimal Forest Growth

Let $N(t)$ represent a population (bacteria in a test tube, people in a city, trees in a forest). The Malthusian model assumes a rate of growth proportional to the population: $N'(t) = aN(t)$. This gives the exponential growth law $N(t) = e^{at}N(0)$, which is only accurate for relatively small values of $N(t)$; overcrowding and competition for resources lower the rate of growth. A more realistic model assumes a steadily decreasing, eventually negative growth coefficient $a(N)$. Assuming $a(N)$ linear, Verhulst's **logistic equation**

$$N'(t) = (a - bN(t))N(t) \tag{1.8.1}$$

results, where $a, b > 0$. This model gives good results for bacteria populations, but does not describe accurately phenomena such as forest growth. In fact, the inhibiting effects of new trees on the growth rate are negligible until these have reached a certain "adult" size. Thus, the growth rate should be a function not of $N(t)$ but of $N(t - h)$ for a suitable time delay $h > 0$, leading to the **delayed logistic equation**

$$N'(t) = (a - bN(t - h))N(t). \tag{1.8.2}$$

Similar delay effects are observed in the influence of overcrowding in human populations (for an elementary exposition of logistic equations with and without delays see Haberman [1977, p. 119ff]). Equation (1.8.2) (the same as (1.8.1)) has two equilibrium solutions: one is $N(t) \equiv 0$, the other

$$N(t) \equiv N_e = a/b. \tag{1.8.3}$$

Assume tree seeds are planted, and trees are logged with seeding and logging rates $u_0(t)$ and $u_1(t)$ respectively. Let k be the time it takes a seed to become a baby tree. Then the equation becomes

$$N'(t) = (a - bN(t-h))N(t) + cu_0(t-k) - u_1(t) \tag{1.8.4}$$

where the coefficient c $(0 \le c \le 1)$ accounts for the fraction of seeds that actually result in a tree. To start the equation we need to know the forest population in an interval of length h,

$$N(t) = N_0(t) \quad (t_0 - h \le t \le t_0). \tag{1.8.5}$$

To "attain the equilibrium population N_e" has at least two meanings. One is

$$N(\bar{t}) = N_e \tag{1.8.6}$$

and says the population is at equilibrium at $t = \bar{t}$ but not necessarily afterwards. If the population is to stay at equilibrium, the target condition must be

$$N(t) = N_e \quad (\bar{t} - h \le t \le \bar{t}), \tag{1.8.7}$$

which guarantees that $N(t) = N_e$ for all $t \ge \bar{t}$ if $cu_0(t-k) - u_1(t) = 0$ for $t \ge \bar{t}$. Target conditions of the form (1.8.6) are called **Euclidean**; those of the form (1.8.7) are called **functional**. To see the reason for this name, consider for instance the space $C[-h, 0]$ of continuous functions defined in the interval $-h \le t \le 0$. Given a function $y(t)$, denote by $y_t(\cdot)$ the **section** of $y(\cdot)$ defined by

$$y_t(\tau) = y(t + \tau) \quad (-h \le \tau \le 0). \tag{1.8.8}$$

Then the target condition (1.8.7) can be written as an ordinary target condition in the space $C[-h, 0]$:

$$N_{\bar{t}}(\cdot) = N_e \tag{1.8.9}$$

where N_e denotes the constant function. An "optimal net profit" problem is, for instance, to maximize the functional

$$J(t, u_1, u_2) = \alpha \int_0^t u_1(\tau)d\tau - \beta \int_0^t u_0(\tau)d\tau$$

with $\alpha, \beta > 0$ at some fixed time $\bar{t} > 0$; the first term represents the profit from logging and the second, the cost of seeding. Clearly, $u_0(t), u_1(t)$ are nonnegative and it is reasonable to include upper bounds on both rates:

$$0 \le u_0(t) \le R, \qquad 0 \le u_1(t) \le S. \tag{1.8.10}$$

Straight maximization of the profit may result in destruction of the forest at time $\bar{t}$, thus we supplement the problem with a (functional) target condition, say

$$|N(t) - N_e| \le \varepsilon \quad (\bar{t} - h \le t \le \bar{t}) \tag{1.8.11}$$

(N_e the equilibrium solution (1.8.3)), and terminate seeding at time $\bar{t} - k$. If the equilibrium position is stable, this means the forest population will stay near equilibrium after $\bar{t}$. Admissible controls for this problem are pairs $(u_0(t), u_1(t))$, u_0 defined in $-k \le t \le \bar{t}$, u_1 defined in $0 \le t \le \bar{t}$, and satisfying (1.8.10) in their respective intervals of definition.

Another growth model is described by the integrodifferential equation

$$N'(t) = \left(a - \left(\int_{-h}^{0} b(\tau)N(t+\tau)d\tau\right)\right)N(t) + cu_0(t-k) - u_1(t) \tag{1.8.12}$$

which takes into account the inhibiting effects of new trees of all sizes on the growth rate.

1.9. Control of Partial Differential Equations: Optimal Cooling of a Plate and Optimal Stabilization of a Vibrating Membrane

Consider a plate occupying a domain Ω with boundary Γ in 2-dimensional Euclidean space $\mathbb{R}^2$. In suitable units, the nonlinear partial differential equation

$$\frac{\partial y(t,x)}{\partial t} = \Delta y(t,x) - f(y(t,x)) + u(t,x) \quad (x \in \Omega), \tag{1.9.1}$$

$$\frac{\partial y(t,x)}{\partial \nu} = 0 \quad (x \in \Gamma), \tag{1.9.2}$$

($x = (x_1, x_2)$, Δ the Laplacian, $\partial/\partial\nu$ the outer normal derivative on Γ) describes the temperature $y(t,x)$ in Ω. The sum $-f(y(t,x)) + u(t,x)$ means applied heat, the first term through feedback, the second as a control, subject to the constraint

$$0 \le u(t,x) \le R. \tag{1.9.3}$$

Condition (1.9.2) indicates the boundary is insulated (there is no heat flow through the boundary). A conceivable problem is to drive the temperature from an initial value

$$y(0,x) = \zeta(x) \tag{1.9.4}$$

to a final value

$$y(\bar{t}, x) = \bar{y}(x) \tag{1.9.5}$$

in time $\bar{t}$, minimizing the cost of heating/cooling, which might be measured in the form

$$y_0(t, u) = \int_{(0,t)\times\Omega} u(\tau, x)^2 dx\, d\tau.$$

A number of variants are possible. For instance, control may be a k-dimensional function $(u_1(t), u_2(t), \dots, u_k(t))$ of t entering the equation as $u(t, x) = \Sigma g_j(x)u_j(x)$, or the **exact** or **point** target condition (1.9.5) may be weakened to approximate conditions such as $|y(\bar{t}, x) - \bar{y}(x)| \le \varepsilon \ \ (x \in \Omega)$ or

$$\int_\Omega (y(\bar{t}, x) - \bar{y}(x))^2 dx \le \varepsilon.$$

Control may also be applied on the boundary,

$$\frac{\partial y(t, x)}{\partial \nu} = g(y(t, x), u(t, x)) \quad (x \in \Gamma). \tag{1.9.6}$$

Engineers call systems such as (1.9.1)-(1.9.2) **distributed parameter systems**; when control appears in the boundary condition as in (1.9.6), we have a **boundary control system**.

For a peek into the results, we consider the distributed parameter system (1.9.1) with boundary condition (1.9.2), control constraint (1.9.3), fixed terminal time $\bar{t}$, and no target condition. The cost functional measures the final deviation from the target:

$$y_0(t, u) = \int_\Omega (y(\bar{t}, x) - \bar{y}(x))^2 dx. \tag{1.9.7}$$

We do spike perturbations in time,

$$u_{s,h,v}(t, x) = \begin{cases} v(x) & (s - h < t \le s) \\ u(t, x) & \text{elsewhere} \end{cases} \tag{1.9.8}$$

where $v(\cdot)$ is an element of the (functional) control set U defined by $0 \le v(x) \le R$. If $\bar{u}(t, x)$ is an optimal control,

$$y_0(\bar{t}, \bar{u}_{s,h,v}) \ge y_0(\bar{t}, \bar{u})$$

for s, h, $v(\cdot)$ arbitrary; thus, if

$$\xi_0(t) = \frac{d}{dh}\bigg|_{h=0+} y_0(t, \bar{u}_{s,h,v}) = \lim_{h\to 0+} \frac{1}{h}\big(y_0(t, \bar{u}_{s,h,v}) - y_0(t, \bar{u})\big), \tag{1.9.9}$$

we have

$$\xi_0(\bar{t}) \geq 0 \tag{1.9.10}$$

for s, $v(\cdot)$ arbitrary. As in **1.6**, the first step in the computation of $\xi_0(t)$ is to figure out

$$\begin{aligned}\xi(t,x) &= \frac{d}{dh}\bigg|_{h=0+} y(t,x,u_{s,h,v}) \\ &= \lim_{h\to 0+} \frac{1}{h}\big(y(t,x,u_{s,h,v}) - y(t,x,u)\big),\end{aligned} \tag{1.9.11}$$

where $y(t,x,u)$ indicates the solution of (1.9.1)-(1.9.2) corresponding to $u = u(t,x)$. A formal computation similar to that in the lines following (1.6.5) reveals that $\xi(t,x)$ is the solution of the linear initial value problem

$$\begin{aligned}\frac{\partial\xi(t,x)}{\partial t} &= \Delta\xi(t,x) - \frac{\partial f(y(t,\bar{u},x))}{\partial y}\xi(t,x) \\ &\quad + (v(x) - \bar{u}(s,x))\delta(t-s) \quad (0 \leq t \leq \bar{t}, x \in \Omega)\end{aligned} \tag{1.9.12}$$

$$\frac{\partial\xi(t,x)}{\partial\nu} = 0 \quad (0 \leq t \leq \bar{t}, x \in \Gamma). \tag{1.9.13}$$

$$\xi(0,x) = 0 \quad (x \in \Omega) \tag{1.9.14}$$

(δ the Dirac delta) or, equivalently,

$$\frac{\partial\xi(t,x)}{\partial t} = \Delta\xi(t,x) - \frac{\partial f(y(t,\bar{u},x))}{\partial y}\xi(t,x) \quad (s \leq t \leq \bar{t}, x \in \Omega), \tag{1.9.15}$$

$$\frac{\partial\xi(t,x)}{\partial\nu} = 0 \quad (s \leq t \leq \bar{t}, x \in \Gamma), \tag{1.9.16}$$

$$\xi(s,x) = v(x) - \bar{u}(s,x) \quad (x \in \Omega). \tag{1.9.17}$$

We then compute $\xi_0(t)$ and use (1.9.10), obtaining

$$\int_\Omega (y(\bar{t},\bar{u},x) - \bar{y}(x))\xi(\bar{t},x)dx \geq 0, \tag{1.9.18}$$

where $0 < s \leq t$ and $v(\cdot) \in U$. Now, let $z(t,x)$ be the solution of the backwards equation

$$\frac{\partial z(t,x)}{\partial t} = -\Delta z(t,x) + \frac{\partial f(y(t,\bar{u},x))}{\partial y}z(t,x) \quad (0 \leq t \leq \bar{t}, x \in \Omega) \tag{1.9.19}$$

$$\frac{\partial z(t,x)}{\partial\nu} = 0 \quad (0 \leq t \leq \bar{t}, x \in \Gamma), \tag{1.9.20}$$

$$z(\bar{t},x) = y(\bar{t},\bar{u},x) - \bar{y}(x) \quad (x \in \Omega). \tag{1.9.21}$$

Then

$$\begin{aligned}
&\int_\Omega z(\bar{t}, x)\xi(\bar{t}, x)dx - \int_\Omega z(s, x)\xi(s, x)dx \\
&= \int_{(s,\bar{t})\times\Omega} \frac{\partial}{\partial t}\big(z(t, x)\xi(t, x)\big)dxdt \\
&= \int_{(s,\bar{t})\times\Omega} \left(\frac{\partial z(t, x)}{\partial t}\xi(t, x) + z(t, x)\frac{\partial \xi(t, x)}{\partial t}\right)dx\,dt = 0
\end{aligned}$$

in view of the divergence theorem. Accordingly, (1.9.18) is

$$\int_\Omega z(s, x)(v(x) - \bar{u}(s, x))dx \geq 0$$

for all s, $v(\cdot)$, or

$$\int_\Omega z(s, x)\bar{u}(s, x) = \min_{v(\cdot)\in U} \int_\Omega z(s, x)v(x)dx \tag{1.9.22}$$

for $0 < s \leq 1$, another protoexample of Pontryagin's minimum principle. Due to the control constraint, (1.9.22) implies

$$\bar{u}(s, x) = \begin{cases} R & \text{where } z(s, x) < 0, \\ 0 & \text{where } z(s, x) > 0 \end{cases} \tag{1.9.23}$$

but gives no information on $u(s, x)$ in the set $e \subseteq (0, \bar{t}) \times \Omega$ where $z(t, x) = 0$. If $y(\bar{t}, \bar{u}, x) \neq y(\bar{t})$ then $z(s, x)$ is nontrivial (that is, not identically zero), but there is some distance from this property to the statement that e has measure zero; when it does, (1.9.23) gives information on the optimal control $\bar{u}(t, x)$ almost everywhere and deserves to be called a **bang-bang principle**. For more on this, see Chapter 11, in particular Problem 11.6.7 and the Miscellaneous Comments to Part II.

Typically for the maximum principle, (1.9.22) does not determine $\bar{u}$ directly; in fact, both the equation (1.9.19) for $z(s, x)$ and the final condition (1.9.21) presuppose knowledge of the unknown optimal solution $y(s, x, \bar{u})$. Even in the linear case, where (1.9.19) does not depend on the optimal solution, we must know $y(\bar{t}, \bar{u}, x)$ in (1.9.21). We shall take up the study of optimal control problems described by parabolic equations in Part II of this work.

Another problem with some claims to realism is that of bringing to equilibrium a vibrating membrane occupying the domain Ω and glued to the boundary; in suitable units, the corresponding system is

$$\frac{\partial^2 y(t, x)}{\partial t^2} = \Delta y(t, x) - f(y(t, x)) + u(t, x) \quad (x \in \Omega), \tag{1.9.24}$$

$$y(t, x) = 0 \quad (x \in \Gamma), \tag{1.9.25}$$

where the sum $-f(y(t,x)) + u(t,x)$ represents an applied force, the first term through feedback, the second as a control subject to a constraint, for instance

$$|u(x,t)| \leq R \tag{1.9.26}$$

or

$$\int_\Omega u(x,t)^2 dx \leq R. \tag{1.9.27}$$

Bringing the membrane to equilibrium from the initial conditions

$$y(0,x) = \zeta_0(x), \qquad \frac{\partial y(0,x)}{\partial t} = \zeta_1(x) \tag{1.9.28}$$

in time $\bar{t}$ amounts to the target condition(s)

$$y(\bar{t},x) = 0, \qquad \frac{\partial y(\bar{t},x)}{\partial t} = 0, \tag{1.9.29}$$

and the membrane will stay at equilibrium if the restoring force $f(y)$ satisfies $f(0) = 0$ and application of the control force terminates at $t = \bar{t}$. As in the cooling problem, control could be finite dimensional or applied through the boundary condition; the *distributed parameter* or *boundary* labels apply.

The two control problems above do not include state constraints for reasons of simplicity, but more realistic modeling must take them into account. For instance, equation (1.9.1) can only be expected to describe the evolution of the temperature $y(t,x)$ in an actual heating/cooling process within a certain range, which justifies the first of the bounds

$$|y(t,x)| \leq K, \qquad |\nabla y(t,x)| \leq L. \tag{1.9.30}$$

The second restriction reflects the fact that, in cooling a material such as glass, large temperature gradients may produce cracks and should be avoided. Of course, constraints such as (1.9.30) make the problem much harder to handle. However, we shall see in Chapter 11 that the minimum principle (1.9.22) still holds with a different definition of $z(t,x)$.

In the vibration model (1.9.24), the integral state constraint

$$\frac{1}{2}\int_\Omega \left\{ \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial x}\right)^2 \right\} dx \leq E^2 \tag{1.9.31}$$

puts a bound on the energy and reminds us of the fact that the wave equation (1.9.24) is just an approximation to the "true" nonlinear equation describing the vibration of the membrane, and that this approximation is only valid at low energy levels.

1.10. Finite Dimensional and Infinite Dimensional Control Problems

There is an important difference between the landing problem in **1.4** and the optimal control problems in **1.8** and **1.9**. In the former, as well as in any other optimal control problem that fits into the ordinary differential model in **1.5**, the **state** of the system is a finite dimensional vector; in the soft landing problem, this vector is the 3-dimensional vector $(h(t), v(t), m(t))$.

The state of the system described by the delay differential equation (1.8.4) (or the functional differential equation (1.8.12)) is a finite dimensional vector as well, namely the 1-dimensional vector $N(t)$. This way to look at the equation is adequate if one deals with the Euclidean target condition (1.8.6). However if the target condition is functional like (1.8.7) it is natural to consider as state of the system at time t the section y_t, which belongs to the infinite dimensional space $C[-h, 0]$.

The state of the system (1.9.1)-(1.9.2) at time t is a function $y(t, \cdot)$, thus an element of an infinite dimensional function space, for example $C(\overline{\Omega})$ or $L^2(\Omega)$. Same for the system (1.9.24)-(1.9.25): the state of the system at time t is the vector $(y(t, \cdot), y_t(t, \cdot))$ in a suitable energy space such as $H_0^1(\Omega) \times L^2(\Omega)$.

Ordinary differential equations like (1.5.2) and partial differential equations like (1.9.10) or (1.9.24) are specimens with something in common: all are *evolution equations*, that is, they describe a system's evolution in time. It comes as no surprise that their control theory contains many common elements, a thread running through this work. However, the treatment of systems whose states lie in an infinite dimensional space is much more involved, and complete generalizations of finite dimensional results are often unavailable.