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H. O. Fattorini
Excerpt
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Part I

Finite Dimensional Control Problems

1

Calculus of Variations and Control Theory

1.1. Calculus of Variations: Surface of Revolution of Minimum Area

Consider the surfaces Σ of revolution about the x-axis whose boundary consists of the circles

$$\Gamma_a = \{(x, y, z); x = a, y^2 + z^2 = r_a\}$$
$$\Gamma_b = \{(x, y, z); x = b, y^2 + z^2 = r_b\}$$

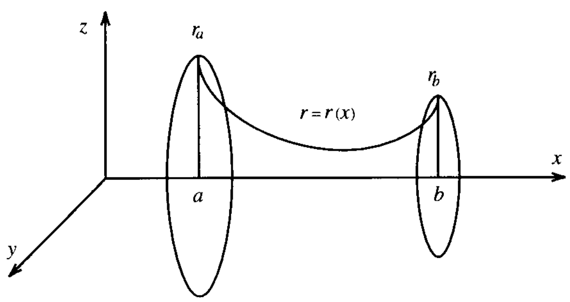

Figure 1.1.

($a < b$, $r_a, r_b \geq 0$). Among these surfaces, we look for one having minimum area. If $r = r(x) \geq 0$ ($a \leq x \leq b$) is the equation of the surface, the area is[(1)]

$$A(r) = 2\pi \int_a^b r(x)\sqrt{1 + r'(x)^2}dx.$$

[(1)] If $r(x)$ is negative for some x the expression for the area is incorrect unless one replaces $r(x)$ by $|r(x)|$ in the integrand. This is ignored in some textbook treatments.

Since Γ_a and Γ_b are the boundary of Σ,

$$r(a) = r_a, \qquad r(b) = r_b. \tag{1.1.1}$$

More generally, we may minimize

$$J(y) = \int_a^b F(x, y(x), y'(x))dx. \tag{1.1.2}$$

The "variable" in J is itself a function $y(x)$ defined in the interval $a \le x \le b$. An expression like this is called a **functional**; the **admissible** functions $y(x)$ in the integrand belong to the space $C^{(1)}[a, b]$ of continuously differentiable functions in $a \le x \le b$ and satisfy boundary conditions, in this case (1.1.1). Admissible functions are an *affine* subspace of $C^{(1)}[a, b]$: if $y(x)$ satisfies (1.1.1) and $v(x)$ belongs to the subspace $C_0^{(1)}[a, b]$ of $C^{(1)}[a, b]$ defined by $v(a) = v(b) = 0$ then

$$y(x) + hv(x) \tag{1.1.3}$$

is admissible for any h. This makes viable the argument below, where we assume that $F(x, y, y')$ is everywhere defined and continuously differentiable with respect to y and y', with F, $\partial F/\partial y$, $\partial F/\partial y'$ continuous in all variables.

Assume that $\bar{y}(\cdot) \in C^{(1)}[a, b]$ is a **minimizing element** or a **minimum** of $J(y)$ (that is, $J(\bar{y}) \le J(y)$ for all admissible $y(\cdot)$). Let $v(\cdot) \in C_0^{(1)}[a, b]$. Then

$$J(\bar{y} + hv) \ge J(\bar{y})$$

for all real h, hence $\phi(h) = J(\bar{y} + hv)$ has a minimum at $h = 0$. This implies $\phi'(0) = 0$. This condition can be written

$$\begin{aligned}
\phi'(0) &= \frac{d}{dh}\bigg|_{h=0} \phi(h) = \frac{d}{dh}\bigg|_{h=0} J(\bar{y} + hv) \\
&= \frac{d}{dh}\bigg|_{h=0} \int_a^b F(x, \bar{y}(x) + hv(x), \bar{y}'(x) + hv'(x))dx \\
&= \int_a^b \left\{ \frac{\partial F}{\partial y}(x, \bar{y}(x), \bar{y}'(x))v(x)dx + \frac{\partial F}{\partial y'}(x, \bar{y}(x), \bar{y}'(x))v'(x) \right\} dx = 0
\end{aligned} \tag{1.1.4}$$

for any $v(\cdot) \in C_0^{(1)}[a, b]$.

Lemma 1.1.1. *Let $f(x)$, $g(x)$ be continuous in $a \le x \le b$. Assume that for every $v \in C_0^{(1)}[a, b]$ we have*

$$\int_a^b \{f(x)v(x) + g(x)v'(x)\}dx = 0. \tag{1.1.5}$$

Then (after possible modification in a null set) $g(\cdot) \in C^{(1)}[a, b]$ and $g'(x) \equiv f(x)$.

Proof. Assume first that $f(x) \equiv 0$; (1.1.5) and the boundary conditions imply

$$\int_a^b (g(x) - c)v'(x)dx = 0 \qquad (1.1.6)$$

for any c. Define

$$v(x) = \int_a^x (g(\xi) - c_0)d\xi$$

where c_0 is such that $v(b) = 0$; then $v(\cdot) \in C_0^{(1)}[a, b]$. Replacing this particular $v(\cdot)$ and c_0 in (1.1.6) we obtain $g(x) \equiv c_0$. In the general case, define

$$F(x) = \int_0^x f(\xi)\, d\xi$$

and integrate (1.1.5) by parts, obtaining

$$\int_a^b \{g(x) - F(x)\}v'(x)dx = 0,$$

for all $v(\cdot) \in C_0^{(1)}[a, b]$ so that $g(x)$ and $F(x)$ differ by a constant. ■

Using this result in (1.1.4) we deduce that $\partial F(x, \bar{y}(x), \bar{y}'(x))/\partial y'$ is a continuously differentiable function of x and that $y(x) = \bar{y}(x)$ satisfies the **Euler equation**

$$\frac{\partial F}{\partial y}(x, y(x), y'(x)) - \frac{d}{dx}\frac{\partial F}{\partial y'}(x, y(x), y'(x)) = 0. \qquad (1.1.7)$$

Hence (assuming J has a minimum in $C^{(1)}[a, b]$), the minimization problem reduces to solving (1.1.7) with boundary conditions (1.1.1). However, the theory of boundary value problems for differential equations is not as simple as that of initial value problems (where the value of a solution and its derivative are given at a single point). For a glimpse on boundary value problems see Elsgolts [1970, p. 165] or Gelfand–Fomin [1963, p. 16]; for a more complete treatment, see Keller [1968].

If $y(\cdot)$ is twice continuously differentiable we may apply the chain rule in the right side of (1.1.7) and obtain a *bona fide* second order differential equation for $\bar{y}(\cdot)$ (however, $y(x)$ may not be so smooth; see Example 1.1.3). If $y(\cdot) \in C^{(2)}[a, b]$ and, in addition, $F(x, y, y') = F(y, y')$ is independent of x, we multiply (1.1.7) by $y'(x)$ and integrate, obtaining

$$F(y(x), y'(x)) - y'(x)\frac{\partial F}{\partial y'}(y(x), y'(x)) = \beta, \qquad (1.1.8)$$

where β is a constant. Conversely, any solution of (1.1.8) with $y'(x) \neq 0$ necessarily satisfies the Euler equation.

The Euler equation for the minimal area problem is

$$\sqrt{1+r'^2} - \frac{d}{dx}\frac{rr'}{\sqrt{1+r'^2}} = 0.^{(2)} \tag{1.1.9}$$

Equation (1.1.8) is

$$r(x) = \beta\sqrt{1+r'(x)^2} \tag{1.1.10}$$

with solution $r(x) \equiv 0$ for $\beta = 0$, and

$$r(x) = \beta\cosh\left(\frac{x-\alpha}{\beta}\right) \tag{1.1.11}$$

for $\beta \neq 0$, where α is arbitrary. (Gelfand–Fomin [1963, p. 20]).

Example 1.1.2. There exist either two, one, or no $r(x)$ of the form (1.1.11) satisfying the boundary conditions (1.1.1). For a proof, see Bliss [1925, p. 90] or Cesari [1983, p. 143]. We just take a look at the case $a = 0$, $b = 1$, $r_a = r_b = r$. To satisfy the boundary conditions we must make $\alpha = 1/2$ and solve the transcendental equation

$$\phi(\beta) = \beta\cosh\left(\frac{1}{2\beta}\right) = r. \tag{1.1.12}$$

If m is the minimum of the function of $\beta > 0$ on the left-hand side, then the equation has no solution if $r < m$, one solution if $r = m$, and two solutions if $r > m$. We have $m = 0.7544\ldots$, attained at $\beta = 0.4167\ldots$.

For $r_a = r_b = 1$ (1.1.12) has two solutions, $\beta = 0.2350\ldots$ and $\beta = 0.8483\ldots$.

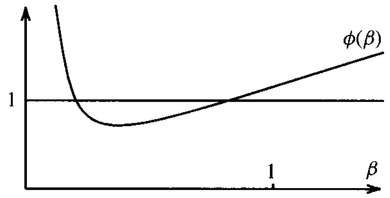

Figure 1.2.

Using formula (1.1.11) the area integral reduces to

$$2\pi\beta\int_0^1 \cosh^2((2x-1)/2\beta)dx = \pi\beta + \pi\beta^2\sinh(1/\beta).$$

The surface corresponding to $\beta = 0.2350\ldots$ (resp. to $\beta = 0.8483\ldots$) has area $6.8456\ldots$ (resp. $5.9918\ldots$).

(2) A minimizing element $\bar{r}(x)$ of $A(r)$ satisfies the Euler equation (1.1.9) if $\bar{r}(x) > 0$; otherwise, $\bar{r}(x) + hv(x)$ may be zero for h arbitrarily small invalidating (1.1.4). Note also that no solution of (1.1.9) may be zero anywhere.

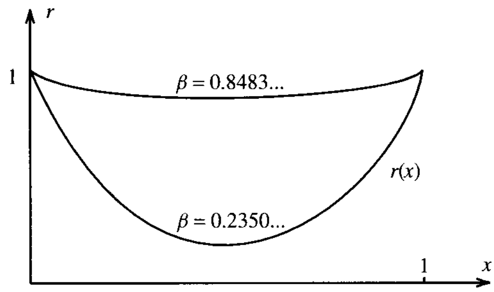

Figure 1.3.

Obviously, the first surface cannot be a minimum. The second is, although this is far from obvious; in fact, it is not even clear whether a minimal surface exists. On the other hand, if $r_a = r_b = m$, the only solution of the form (1.1.11) satisfying the boundary conditions (1.1.1), whose area is $4.2903\ldots$ is not a minimum of the functional. In fact, the "surface" consisting of the two disks spanned by Γ_0 and Γ_1 connected by the segment of the x-axis between them has area $2\pi\beta^2\cosh^2(1/2\beta) = 3.5762\ldots < 4.2903\ldots$. Obviously, this is not one of the surfaces allowed to compete for the minimum, but it can be approximated by smooth surfaces of revolution having almost the same area, for instance, $r_n(x) = (x^n + (1-x)^n)\beta\cosh(1/2\beta)$ for large n (see Figure 1.4).

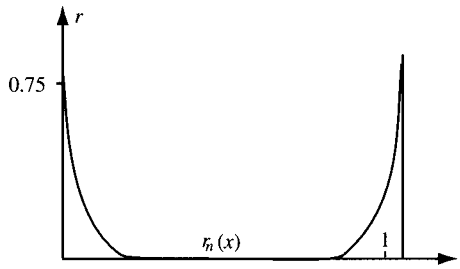

Figure 1.4.

Taking $r_a = r_b = m' > m$ with m' sufficiently close to m we obtain two functions of the form (1.1.11) satisfying the boundary conditions, none of which is a minimum.

For a complete solution of the minimal surface problem in the spirit of control theory, see **10.5**; classical treatments are given in Bliss [1925, Ch. IV] or Cesari [1983, p.143].

The results on the functional (1.1.2) extend easily to functionals depending on n functions $y_1(x), \ldots, y_n(x)$ and their derivatives,

$$J(y_1, \ldots, y_n) = \int_a^b F(x, y_1(x), \ldots, y_n(x), y_1'(x), \ldots, y_n'(x))dx. \quad (1.1.13)$$

Dealing with each $y_j(x)$ separately, we deduce that if F is everywhere defined and continuously differentiable with respect to each y_j and y_j' with F and partial derivatives continuous in all arguments, then a minimum $\bar{y}_1(x), \dots, \bar{y}_n(x)$ of (1.1.11) where each $\bar{y}_j(x)$ belongs to $C^{(1)}[a, b]$ must satisfy the **Euler equations**

$$\frac{\partial F}{\partial y_j} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_j'}\right) = 0 \quad (j = 1, 2, \dots, n). \tag{1.1.14}$$

Example 1.1.3. (Gelfand–Fomin [1963, p. 16]) The functional

$$J(y) = \int_{-1}^{1} y^2(x)\big(2x - y'(x)\big)^2 dx$$

with boundary conditions $y(-1) = 0$, $y(1) = 1$ attains its minimum (zero) for $\bar{y}(x) = 0 \ \ (x \le 0)$, $\bar{y}(x) = x^2 \ \ (x \ge 0)$. The minimum $\bar{y}(\cdot)$ does not belong to $C^{(2)}[a, b]$.

Example 1.1.4. (Gelfand–Fomin [1963, p. 17]) Assume $F(x, y, y')$ has continuous partials up to order two in all variables. Let $\bar{y}(\cdot) \in C^{(1)}[a, b]$ be a solution of Euler's equation (1.7) with

$$\frac{\partial^2 F}{\partial y'^2}(x, \bar{y}(x), \bar{y}'(x)) \ne 0 \quad (a \le x \le b). \tag{1.1.15}$$

Then $\bar{y}(\cdot) \in C^{(2)}[a, b]$. This applies to the minimal area problem (where $\partial^2 F(r, r')/\partial r'^2 = r(1 + r'^2)^{-3/2}$) as follows: if $r = r(x)$ is the equation of a minimal surface in $C^{(1)}[a, b]$ with $r(x) > 0$, then $r(\cdot) \in C^{(2)}[a, b]$.

1.2. Interpretation of the Results

All we have shown on the problem of minimizing (1.1.2) is that if $\bar{y}(\cdot) \in C^{(1)}[a, b]$ is a minimum, then $\bar{y}(\cdot)$ satisfies the Euler equation (1.1.7). Thus, we only have *necessary* conditions for a minimum. They may not be sufficient: a solution of (1.1.7) satisfying the boundary conditions may not be a minimum of $J(y)$, as we have seen in Example 1.1.2. We meet the same problem in calculus trying to find the minima of a function $f(x) = f(x_1, x_2, \dots, x_m)$ in m-dimensional Euclidean space $\mathbb{R}^m$: at a minimum $\bar{x} = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m)$ of f we have

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_2} = \cdots = \frac{\partial f}{\partial x_m} = 0 \tag{1.2.1}$$

but these conditions are not sufficient. Points $\bar{x} \in \mathbb{R}^m$ where (1.2.1) holds are called **extremals** of the function f, and we use the same terminology for the functional (1.1.2): a function $y \in C^{(1)}[a, b]$ satisfying (1.1.7) and the boundary conditions is an **extremal** of J. For instance, the inner surface in Figure 1.3 for $r_a = r_b = 1$ is an extremal but not a minimum.

In some cases, necessary conditions in combination with existence theorems give the actual minima of a functional. For instance, if a minimizing element $\bar{y}(\cdot) \in C^1[a, b]$ exists and solutions of the boundary value problem are unique, then the solution of the boundary value problem must be the minimum. However, this may fail as seen in Example 1.1.2 for the minimal area problem; solutions in $C^{(1)}[a, b]$ may not exist or the boundary value problem may have multiple solutions. Another problem without smooth solutions is

Example 1.2.1. (Gelfand–Fomin [1963, p. 61]) The functional

$$J(y) = \int_{-1}^{1} y^2(x)(1 - y'(x))^2 dx$$

with boundary conditions $y(-1) = 0$, $y(1) = 1$ attains its minimum (zero) for $\bar{y}(x) = 0 \quad (x \le 0)$, $\bar{y}(x) = x \quad (x \ge 0)$. The minimum $\bar{y}(\cdot)$ does not belong to $C^{(1)}[a, b]$.

Proper treatment of variational problems (and of control problems) needs a less demanding definition of solution; see **10.5** for more on this.

1.3. Mechanics and Calculus of Variations

Consider a mechanical system with a finite number of degrees of freedom. We denote by $q_1, q_2, \ldots, q_n$ the **generalized coordinates** of the system, in terms of which the Cartesian coordinates can be determined in 3-space. The n-dimensional point $q = (q_1, q_2, \ldots, q_n)$ moves arbitrarily in a region of Euclidean space $\mathbb{R}^n$ or, more generally, in an n-dimensional differential manifold. Assume for simplicity that the system consists of a finite number of particles with Cartesian coordinates $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_p \in \mathbb{R}^3$:

$$\mathbf{r}_j = \mathbf{r}_j(q_1, \ldots, q_n) \quad (1 \le j \le p), \tag{1.3.1}$$

and that the forces acting on the system are due to a potential,

$$F_j = -\nabla_{\mathbf{r}_j} U(\mathbf{r}_1, \ldots, \mathbf{r}_j, \ldots, \mathbf{r}_p) \quad (j = 1, 2, \ldots, p).$$

The Lagrangian of the system is

$$L = \sum_{j=1}^{p} \frac{m_j}{2} \|\mathbf{r}_j'\|^2 - U(\mathbf{r}_1, \ldots, \mathbf{r}_p) = T - U \tag{1.3.2}$$

where the **kinetic energy** T and the **potential energy** U are expressed in terms of the generalized coordinates. The motion of the system is described by **Hamilton's**

principle (Kompaneyets [1978, p. 17]): the possible motions $q_1(t), q_2(t), \ldots, q_n(t)$ of the system in a time interval $t_0 \le t \le t_1$ are extremals of the **action integral**

$$S = \int_{t_0}^{t_1} L(q_1, q_2, \ldots, q_n, q_1', q_2', \ldots, q_n')dt, \tag{1.3.3}$$

that is, they solve the Euler equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_j'}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{1.3.4}$$

((see (1.1.14)). These are the **Euler-Lagrange equations** of mechanics and are combined with initial or boundary conditions: usually, the initial position and velocity of the system (that is, the position and velocity at $t = t_0$) are given. It also makes sense to specify the position of the system at different times t_0 and t_1, which produces a boundary value problem in the sense of **1.1.**

Example 1.3.1. A **simple pendulum** is a particle of mass m connected to the origin by a rigid rod of length l and allowed to move on the (x, y)-plane.

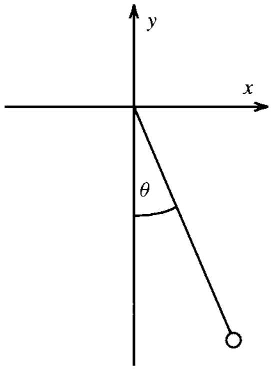

Figure 1.5.

Only one generalized coordinate, the angle θ, is necessary. It moves in the circle obtained identifying points modulo 2π on the line. The force of gravity comes from the potential energy $U = mgy$ (g the acceleration of gravity). Since $\mathbf{r} = l(\sin\theta, -\cos\theta)$, $\mathbf{r}' = l\theta'(\cos\theta, \sin\theta)$ and

$$L = T - U = \frac{m}{2}l^2\theta'^2 - mgl(1 - \cos\theta), \tag{1.3.5}$$

(where we have taken arbitrarily the stable equilibrium position as having potential energy zero). The Euler-Lagrange equation is the nonlinear pendulum equation

$$\theta'' + \frac{g}{l}\sin\theta = 0. \tag{1.3.6}$$

1.4. Optimal Control: Fuel Optimal Landing of a Space Vehicle

A space vehicle on a vertical trajectory tries to land smoothly (that is, with velocity zero) on the surface of a planet (see Figure 1.6). Denote by $h(t)$ the height at time t (so that $v(t) = h'(t)$ is the instantaneous velocity). Since combustible is being consumed, the mass $m(t)$ of the vehicle is a nonincreasing function of t. If we call $u(t)$ the instantaneous upwards thrust, Newton's law gives $m(t)h''(t) = -gm(t) + u(t)$, where g is the acceleration of gravity. Assuming that the thrust is proportional to the rate of decrease of mass (that is, proportional to the rate at which combustible is used up) we introduce $v(t) = h'(t)$ as a variable and obtain the following first-order system of differential equations:

$$h'(t) = v(t), \qquad v'(t) = -g + \frac{u(t)}{m(t)}, \qquad m'(t) = -Ku(t),$$

where $K > 0$. At the initial time $t_0 = 0$ we have initial conditions

$$h(0) = h_0, \qquad v(0) = v_0, \qquad m(0) = m_0.$$

The vehicle will land softly at time $\bar{t} \geq 0$ if

$$h(\bar{t}) = 0, \qquad v(\bar{t}) = 0.$$

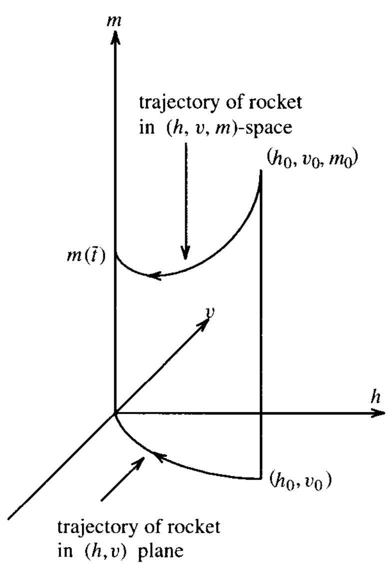

Figure 1.6.

The thrust cannot be negative or arbitrarily large:

$$0 \leq u(t) \leq R$$