The Geometry of Total Curvature on Complete Open Surfaces

This is a self-contained account of how some modern ideas in differential geometry can be used to tackle and extend classical results in integral geometry. The authors investigate the influence of the total curvature on the metric structure of complete noncompact Riemannian 2-manifolds, though their work, much of which has never appeared in book form before, can be extended to more general spaces.

Many classical results are introduced and then extended by the authors. The compactification of complete open surfaces is discussed, as are Busemann functions for rays. Open problems are provided in each chapter, and the text is plentifully illustrated with figures designed to improve the reader's intuitive understanding of the subject matter.

The treatment is self-contained, assuming only a basic knowledge of manifold theory, and so is suitable for graduate students and nonspecialists who seek an introduction to this modern area of differential geometry.

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Preface

The study of the curvature and topology of Riemannian manifolds is mainstream in differential geometry. Many of the important contributions in this topic go back to the pioneering works by Cohn-Vossen in 1935–6, [**19**] and [**20**]. In fact the study of total curvature on complete noncompact Riemannian manifolds made by him contains many fruitful ideas. Many hints in his thoughts lead us to the study of the curvature and topology of Riemannian manifolds.

The well-known Gauss–Bonnet theorem states that the total curvature of a compact Riemannian 2-manifold is a topological invariant. Cohn-Vossen first proved that the total curvature of a finitely connected complete noncompact Riemannian 2-manifold M is bounded above by $2\pi\chi(M)$, where $\chi(M)$ is the Euler characteristic of M. Among many beautiful consequences of this result, he proved the splitting theorem for complete open Riemannian 2-manifolds of nonnegative Gaussian curvature admitting a straight line. The structure theorem for such 2-manifolds was also established by him. He investigated the global behavior of complete geodesics on these 2-manifolds and this gave rise to the study of poles. The Bonnesen-type isoperimetric problem for complete open surfaces admitting a total curvature was first investigated by Fiala [**26**] for the analytic case and then by Hartman [**34**] for the C^2 case. Here the Cohn-Vossen theorem plays an essential role. The total curvature of infinitely connected complete open surfaces was discussed by Huber from the point of view of complex analysis. Busemann considered the notion of total curvature on a G-surface X (see Section 43, [**12**]), in which he suggested that Cohn-Vossen's results would follow on Busemann G-surfaces when the total curvature was replaced by the Busemann total excess of X.

It took more than thirty years to obtain higher-dimensional extensions of Cohn-Vossen's results. They are the Toponogov splitting theorem [**103**] and the structure theorem for complete noncompact Riemannian manifolds of positive sectional curvature [**30**] and of nonnegative sectional curvature [**17**]. The total curvature of higher-dimensional complete noncompact Riemannian manifolds of nonnegative sectional curvature was discussed in [**69**].

The purpose of this book is to study the geometric significance of the total curvature of complete noncompact Riemannian 2-manifolds. The total curvature $c(M)$ of such a manifold M is not a topological invariant but is dependent upon the choice of Riemannian metric. Therefore we may consider that $c(M)$ describes certain geometric properties of M. These phenomena are seen in the asymptotic behavior of the mass of rays on M and that of the isoperimetric inequalities for metric balls and their boundaries. Moreover, the size of the ideal boundary equipped with the Tits metric is determined by $c(M)$ and the topology of M. The global behavior of complete geodesics on Riemannian planes is controlled by $c(M)$. It is expected that many results will be extended to complete noncompact Alexandrov surfaces with the Busemann total excess.

This book is written as a self-contained text including many examples, figures and exercises. First-year graduate students will find this book very useful. The reader will quickly acquire the tools necessary for the study of Riemannian geometry.

In Chapter 1, the basic tools in Riemannian geometry are prepared. We first use only local coordinates and introduce the Levi–Civita connection and curvature tensor. We then use vector field notation to simplify the working. We want to acknowledge two books which have been very useful in writing this chapter. The discussion in Sections 1.4 to 1.7 is based on the book by Gromoll, Klingenberg and Meyer [**29**]. The discussion on the Sasaki metric is based on Sakai's book [**73**].

In Chapter 2, the classical results by Cohn-Vossen and Huber on the total curvature of complete open surfaces are introduced. All the ideas employed by Cohn-Vossen in [**19**] and [**20**] are explained here. We deal with the Gauss–Bonnet theorem for compact simplicial complexes on surfaces in such a way that the Gauss–Bonnet theorem can be extended to them.

In Chapter 3, the ideal boundary $M(\infty)$ of a complete noncompact Riemannian 2-manifold M with total curvature is obtained by using the idea of Ballmann, Gromov and Schroeder [**7**], which they discussed using Hadamard manifolds. New ideas are introduced. We establish the Gauss–Bonnet theorem for the compactification $M \cup M(\infty)$ of M by attaching an ideal boundary equipped with the Tits metric. A new triangle comparison theorem is established for special triangles bounding domains having small total absolute curvature. Furthermore, we prove that the scaling limit of M with finite total curvature is the union of flat cones generated by $M(\infty)$ all having a common vertex. The behavior of Busemann functions is discussed.

In Chapter 4, the structure of the cut loci of circles on complete open surfaces with (or without) total curvature is discussed. The classical Hartman theorem on geodesic parallel circles is introduced. The topological structure of metric circles is discussed in detail.

In Chapter 5, the isoperimetric inequalities for metric circles and for balls around a smooth Jordan circle are discussed. The classical Fiala–Hartman theorem is extended. The infinitely connected case is also considered.

In Chapter 6, the mass of rays emanating from a point on M is discussed. Integral formulae for the mass of rays are treated in connection with the isoperimetric inequalities.

In Chapter 7, the classical result due to von Mangoldt is presented. The set of poles on a surface of revolution homeomorphic to a plane is determined explicitly: it consists of a unique trivial pole or forms a closed ball centered at the vertex. We find a necessary and sufficient condition for a surface of revolution to have many poles. The cut locus of a standard surface of revolution, such as a two-sheeted hyperboloid, is determined explicitly.

In Chapter 8, the global behavior of complete geodesics on a Riemannian plane M having a total curvature is discussed. The number of self-intersections of complete geodesics away from a compact set (near the ideal boundary) is estimated explicitly in terms of the total curvature of M. This involves the Whitney regular homotopy of curves and the rotation numbers.

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