1

CLIFFORD ALGEBRAS

In this opening chapter, we collect together fundamental properties of a variety of (complex) Clifford algebras attached to the real inner product space V. The plain complex Clifford algebra $C(V)$ is the universal unital associative complex algebra containing V as a real subspace with the property that if $v \in V$ then $v^2 = \|v\|^2\mathbf{1}$; this algebra has a unique involution such that if $v \in V$ then $v^* = v$. The involutive algebra $C(V)$ carries a unique norm with the C^* property, that if $a \in C(V)$ then $\|a^*a\| = \|a\|^2$; the completion of $C(V)$ relative to this norm is a C^* algebra called the C^* Clifford algebra $C[V]$. The C^* algebra $C[V]$ has a unique (even) state τ with the (central) property that if $a, b \in C[V]$ then $\tau(ba) = \tau(ab)$; the von Neumann algebra generated in the corresponding cyclic representation of $C[V]$ is the vN Clifford algebra $\mathcal{A}[V]$. When V is finite-dimensional, these algebras coincide; when V is infinite-dimensional they are all different.

In §1 we present a detailed account of the plain complex Clifford algebra. We begin by studying the most immediate properties of $C(V)$ when V is arbitrary. We next consider $C(V)$ first when V is finite-dimensional and then when V is more particularly even-dimensional. After this, we approach $C(V)$ when V is infinite-dimensional by means of approximations via subspaces of V having finite (often even) dimension. Lastly, we comment on the structure of $C(V)$ when the dimension of V is odd. All of this material is quite standard and may be extracted from a number of sources; we include it here for completeness and as an introduction.

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In §2 we develop the basic structure of the C^* Clifford algebra $C[V]$ when V is infinite-dimensional. The approach we adopt is only one of a number that are possible; brief references to some alternative approaches are given in the Remarks at the end of this chapter. In keeping with the aim expressed in the Introduction, we develop the fundamentals without the assumption that V be separable; again we refer to the Remarks for a few special comments in case V is separable.

In §3 we study the vN Clifford algebra $\mathcal{A}[V]$ when V is infinite- dimensional. As a matter of detail, we introduce it as the von Neumann algebra that arises by closing (in either operator topology, weak or strong) the range of the left regular representation of $C(V)$ on its Hilbert space completion relative to the inner product determined by its unique central state. Once again, we avoid the assumption that V be separable; if V is separable then $\mathcal{A}[V]$ is a version of the hyperfinite II_1 factor, for more on which see the Remarks.

1.1 Clifford algebras

Our primary aim in this opening section is to develop some of the purely algebraic structure of the complex Clifford algebra over a real inner product space. Accordingly, we make no completeness assumptions on the underlying real inner product space, which we allow to have arbitrary dimension.

Thus, let V be an arbitrary real vector space upon which $(\cdot \mid \cdot)$ is a positive-definite inner product and denote by $\|\cdot\|$ the corresponding norm. By a *Clifford map* on V we shall mean a real-linear map $f : V \to B$ into a unital associative complex algebra B such that if $v \in V$ then $f(v)^2 = \|v\|^2 \mathbf{1}$. In these terms, we define a *complex Clifford algebra* over V to be a unital associative complex algebra A together with a Clifford map $\phi : V \to A$ satisfying the following *universal mapping property*: that if $f : V \to B$ is any Clifford map, then there exists a unique algebra map $F : A \to B$ such that $F \circ \phi = f$.

As we now proceed to show, V always carries a complex Clifford algebra and any two complex Clifford algebras over V are naturally isomorphic.

Existence

We dispose of the existence problem for complex Clifford algebras by means of a standard tensor product construction. Denote by $V^{\mathbb{C}} = \mathbb{C} \otimes V$ the complexification of V: thus, $V^{\mathbb{C}}$ is obtained from V by extending from real to complex scalars. Let $T(V)$ stand for the full tensor algebra

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over $V^{\mathbb{C}}$: thus,

$$T(V) = \bigoplus_{r=0}^{\infty} T^r(V)$$

where $T^0(V) = \mathbb{C}$ and where if $r > 0$ then

$$T^r(V) = \overleftarrow{V^{\mathbb{C}}\otimes} \overset{r}{\cdots} \overrightarrow{\otimes V^{\mathbb{C}}}$$

is the r-fold complex tensor power of $V^{\mathbb{C}}$. Of course, $T(V)$ is a unital associative complex algebra with $\mathbf{1} = 1 \in \mathbb{C} = T^0(V)$ as multiplicative identity. Let $I(V)$ be the bilateral ideal of $T(V)$ generated by the subset

$$\{v \otimes v - (v \mid v)\mathbf{1} : v \in V \subset V^{\mathbb{C}}\}.$$

Finally, let A be the quotient algebra $T(V)/I(V)$ and let $\phi : V \to A$ be the map sending $v \in V \subset V^{\mathbb{C}} = T^1(V)$ to its coset modulo $I(V)$. It is plain both that A is a unital associative complex algebra and that $\phi : V \to A$ is a Clifford map. Now, let $f : V \to B$ be a Clifford map and extend to $f : V^{\mathbb{C}} \to B$ by complex linearity. The universal mapping property for the tensor algebra guarantees that f extends uniquely to an algebra map $T(f) : T(V) \to B$. The assumption that f is a Clifford map ensures that $T(f)$ vanishes on the ideal $I(V)$. Consequently, there exists a unique algebra map $F : A \to B$ such that

$$\begin{tikzcd}
T(V) \arrow[dr, "T(f)"] \arrow[dd] & \\
& B \\
A \arrow[ur, "F"'] &
\end{tikzcd}$$

is a commutative diagram, in which the vertical is the canonical quotient map. It is now evident that F is the unique algebra map from A to B satisfying $F \circ \phi = f$.

Uniqueness

That any two complex Clifford algebras over V are naturally isomorphic follows as usual from the universal mapping property. In fact, let A and A' be complex Clifford algebras over V with Clifford maps $\phi : V \to A$ and $\phi' : V \to A'$. Since $\phi : V \to A$ satisfies the universal mapping property and $\phi' : V \to A'$ is a Clifford map, there is a unique algebra map $\Phi' : A \to A'$ such that $\Phi' \circ \phi = \phi'$. Similarly, there is a unique algebra map $\Phi : A' \to A$ such that $\Phi \circ \phi' = \phi$. Now $\Phi \circ \Phi' : A \to A$ is an algebra map such that

$$\Phi \circ \Phi' \circ \phi = \Phi \circ \phi' = \phi$$

whence $\Phi \circ \Phi'$ is the identity map, on account of the universal mapping property for $\phi : V \to A$ applied to $\phi : V \to A$ itself. Similarly, $\Phi' \circ \Phi : A' \to A'$ is also the identity map. Consequently, Φ and Φ' are mutually inverse algebra isomorphisms.

Having thus established that the real inner product space V carries an essentially unique complex Clifford algebra, we may fix one and with impunity refer to it as the *complex Clifford algebra* $C(V)$ of V. Notice that the *Clifford property*

$$v \in V \quad \Rightarrow \quad \phi(v)^2 = \|v\|^2\mathbf{1}$$

satisfied by the Clifford map $\phi : V \to C(V)$ implies that ϕ is necessarily injective. This being so, we shall feel free to suppress ϕ and to identify V with its image in $C(V)$ whenever convenient.

Theorem 1.1.1 *The complex Clifford algebra $C(V)$ is generated by its real subspace V satisfying the Clifford relations*

$$x, y \in V \quad \Rightarrow \quad xy + yx = 2(x \mid y)\mathbf{1}.$$

Proof The tensor algebra $T(V)$ is of course generated by its real subspace $V \subset V^{\mathbb{C}} = T^1(V)$; as a result, the quotient algebra $C(V) = T(V)/I(V)$ is generated by its own copy of V. The Clifford property of $C(V)$ asserts that $v^2 = \|v^2\|\mathbf{1}$ whenever $v \in V$; the Clifford relations follow at once upon polarization, replacing v by $x + y$ when $x, y \in V$. □

The *Clifford relations* just established have as a particular consequence the following fact: that if $x, y \in V$ then

$$(x \mid y) = 0 \quad \Leftrightarrow \quad xy + yx = 0$$

so that vectors in V are orthogonal if and only if they anticommute as elements of $C(V)$. This is but one manifestation of a theme that will be repeated throughout the course of our study: namely, that geometry in V is reflected by algebra in $C(V)$.

Now the universal mapping property for the complex Clifford algebra has certain standard functorial consequences. Fundamental among these is the fact that isometries between real inner product spaces give rise to homomorphisms between their complex Clifford algebras. Here, if $(\cdot \mid \cdot)$ and $(\cdot \mid \cdot)'$ are inner products on the real vector spaces V and V' then the linear map $g : V \to V'$ is said to be *isometric* in case $(gx \mid gy)' = (x \mid y)$ whenever $x, y \in V$. For the sake of clarity, let us reinstate the canonical embeddings $\phi : V \to C(V)$ and $\phi' : V' \to C(V')$ of the real inner product spaces in their complex Clifford algebras.

Theorem 1.1.2 *If $g : V \to V'$ is an isometric linear map then there exists a unique algebra map $\theta_g : C(V) \to C(V')$ such that*

$$\theta_g \circ \phi = \phi' \circ g.$$

Proof By virtue of its isometric nature, g when followed by the canonical embedding $\phi' : V' \to C(V')$ yields a Clifford map $\phi' \circ g : V \to C(V')$. The universal mapping property for $C(V)$ now provides a unique algebra map $G : C(V) \to C(V')$ with the property that $G \circ \phi = \phi' \circ g$. All that remains is to set θ_g equal to G. □

When we once again suppress the canonical embeddings, this result may be formulated as saying that the linear isometry $g : V \to V'$ extends uniquely to an algebra map $\theta_g : C(V) \to C(V')$.

As usual, we shall let $O(V)$ signify the *orthogonal group* of V: thus, $O(V)$ comprises all isometric real-linear automorphisms of V. As a particular instance of the functorial property in the preceding theorem, each orthogonal transformation $g \in O(V)$ extends uniquely to define an automorphism θ_g of the complex Clifford algebra $C(V)$. We shall follow the custom of referring to θ_g as the *Bogoliubov automorphism* of $C(V)$ induced by g. If also $h \in O(V)$ then each of θ_{gh} and $\theta_g \circ \theta_h$ is an automorphism of $C(V)$ extending gh; it follows that $\theta_{gh} = \theta_g \circ \theta_h$. Thus, we in fact have a group homomorphism

$$\theta : O(V) \to \operatorname{Aut} C(V)$$

representing the orthogonal group by automorphisms of the complex Clifford algebra. This automorphic group representation and its descendants will feature quite prominently in what follows.

One particular Bogoliubov automorphism is of special importance and deserves a separate symbol: we denote by γ the Bogoliubov automorphism θ_{-I} induced by minus the identity; thus γ is the unique automorphism of $C(V)$ sending each element of V to its negative. Since the orthogonal transformation $-I$ has period 2, so also does the automorphism γ; accordingly, we refer to γ as the *grading automorphism* of $C(V)$. The subalgebra $\ker(\gamma - I)$ of $C(V)$ fixed pointwise by γ is called the *even complex Clifford algebra* $C^+(V)$ of V; the complementary subspace $\ker(\gamma + I) \subset C(V)$ on which γ acts as minus the identity is denoted by $C^-(V)$. In keeping with our referring to γ as the grading automorphism, we refer to elements of $C^+(V)$ as being *even* and to elements of $C^-(V)$ as being *odd*.

In addition to its grading, the complex Clifford algebra has a canonical antiautomorphism and a canonical conjugation, their product being a canonical adjoint operation on the complex Clifford algebra. We take each of these in turn.

Let us denote by $C(V)^0$ the algebra opposite to $C(V)$: thus, $C(V)^0$ is $C(V)$ as a set, with precisely the same linear structure but with reversed product, so that the identity map $C(V) \to C(V)^0$ is an antiisomorphism of algebras. It is plain that the canonical inclusion $V \to C(V)^0$ is a Clifford map, this being the suppressed $\phi : V \to C(V)$ followed by the identity map $C(V) \to C(V)^0$. The universal mapping property for $C(V)$ provides a unique algebra homomorphism α from $C(V)$ to $C(V)^0$ restricting to V as the identity; of course, we may view α as an antihomomorphism from the algebra $C(V)$ to itself. The composite $\alpha \circ \alpha : C(V) \to C(V)$ is now an algebra homomorphism restricting to V as the identity and hence coinciding on the whole of $C(V)$ with the identity. Thus α is in fact an antiautomorphism of $C(V)$: indeed, it is the unique antiautomorphism of $C(V)$ that fixes V pointwise. We shall refer to α as the *main antiautomorphism* of the complex Clifford algebra. Incidentally, α arises also as follows: reversal of all tensor products defines an antiautomorphism of the tensor algebra $T(V)$ stabilizing the ideal $I(V)$ and α is the antiautomorphism induced on the quotient $T(V)/I(V) = C(V)$.

Let us denote by $\overline{C(V)}$ the algebra conjugate to $C(V)$: thus, $\overline{C(V)}$ is $C(V)$ as a set, with precisely the same ring structure but with conjugated scalar multiplication, so that the identity map $C(V) \to \overline{C(V)}$ is an antilinear ring isomorphism. The canonical inclusion $V \to \overline{C(V)}$ being a Clifford map, the universal mapping property for $C(V)$ provides a unique algebra homomorphism κ from $C(V)$ to $\overline{C(V)}$ restricting to V as the identity; of course, we may view κ as an antilinear ring homomorphism from $C(V)$ to itself. Being an algebra homomorphism restricting to V as the identity, the composite $\kappa \circ \kappa : C(V) \to C(V)$ is the identity on all of $C(V)$. Thus, κ is an antilinear ring automorphism of $C(V)$: in fact, it is the unique such fixing V pointwise. We shall refer to κ as the *main conjugation* of the complex Clifford algebra, often writing $\overline{a}$ in place of $\kappa(a)$ when $a \in C(V)$. Incidentally, the conjugation of $V^{\mathbb{C}}$ pointwise fixing V extends functorially to a conjugation of $T(V)$ stabilizing $I(V)$; the main conjugation of $C(V)$ is the induced map on the quotient $T(V)/I(V)$.

Now the main antiautomorphism α and the main conjugation κ commute; their product is the unique antilinear antiautomorphism of $C(V)$ restricting to V as the identity. Thus, $\alpha \circ \kappa = \kappa \circ \alpha$ is an involution or adjoint operation: we shall call it the *main involution* of the complex

Clifford algebra and shall denote it by a star, so that if $a \in C(V)$ then

$$a^* = \alpha(\bar{a}) = \overline{\alpha(a)}.$$

In this way, $C(V)$ naturally becomes an algebra with involution, or involutive algebra. As such, it satisfies a further universal mapping property, the statement of which requires a definition: if B is an involutive unital associative complex algebra, then the Clifford map $f : V \to B$ is *self-adjoint* in case $f(v)^* = f(v)$ whenever $v \in V$.

Theorem 1.1.3 *If $f : V \to B$ is a self-adjoint Clifford map then the unique algebra map $F : C(V) \to B$ such that $F \mid V = f$ is involution-preserving.*

Proof Simply note that the set $\{a \in C(V) : F(a)^* = F(a^*)\}$ is a subalgebra of $C(V)$ containing V and recall that V generates $C(V)$ as a complex algebra. □

In this regard, it should be noted that if $g \in O(V)$ then the Bogoliubov automorphism θ_g is involution-preserving and hence an automorphism of $C(V)$ as an involutive algebra; moreover, θ_g commutes with the grading automorphism, the main antiautomorphism and the main conjugation.

After these remarks on complex Clifford algebras in general, we now pay more particular attention to the finite-dimensional situation. Thus, let the real inner product space V be finite-dimensional with $\{v_1, \ldots, v_m\}$ as a specific orthonormal basis. It is notationally convenient to write $\mathbf{m}$ in place of $\{1, \ldots, m\}$. If $S = \{s_1 < \ldots < s_p\}$ is a nonempty subset of $\mathbf{m}$ then we shall put

$$v_S = v_{s_1} \ldots v_{s_p}$$

with the product formed in $C(V)$. By convention, we shall associate the multiplicative identity of $C(V)$ to the empty index: $v_\emptyset = \mathbf{1}$. Notice that v_S is a unitary element of the involutive algebra $C(V)$ whenever $S \subset \mathbf{m}$: on the one hand, vectors in V are self-adjoint in being fixed by the main involution; on the other hand, unit vectors in V have square $\mathbf{1}$ on account of the Clifford property. It turns out that $\{v_S : S \subset \mathbf{m}\}$ is a basis for $C(V)$ as a complex vector space, whence $C(V)$ has complex dimension $2^{|\mathbf{m}|} = 2^m$. Our route towards establishing this fact lies by way of properties of the elements $\{v_S : S \subset \mathbf{m}\}$ that prove rather useful in probing further the structure of $C(V)$.

First, let S and T be subsets of $\mathbf{m}$ having cardinalities $|S|$ and $|T|$ respectively. Repeated application of the Clifford relations shows that

$$v_T v_S = (-1)^{|S|\,|T|} v_S v_T$$

whenever S and T are disjoint. In general, we have the following result.

Theorem 1.1.4 *If $S, T \subset \mathbf{m}$ then $v_T v_S = (-1)^{|S|\,|T|+|S\cap T|} v_S v_T$.*

Proof Put $R = S \cap T$, $S' = S - R$ and $T' = T - R$; indicate cardinalities by the corresponding lower case letters. Note that

$$v_S = \sigma v_R v_{S'}, \qquad v_T = \tau v_R v_{T'}$$

where the signs $\sigma, \tau \in \{+1, -1\}$ arise from the Clifford relations as a result of reordering. The special case recorded before the theorem implies that

$$\begin{aligned} \tau\sigma v_T v_S &= v_R v_{T'} v_R v_{S'} \\ &= (-1)^{rt'} v_R v_R v_{T'} v_{S'} \\ &= (-1)^{rt'+s'r+s't'} v_R v_{S'} v_R v_{T'} \\ &= (-1)^{rt'+s'r+s't'} \sigma\tau v_S v_T. \end{aligned}$$

Moreover,

$$rt' + s'r + s't' = (r + s')(r + t') - r^2$$

is congruent to

$$(r + s')(r + t') + r = st + r$$

modulo 2. Consequently,

$$v_T v_S = (-1)^{st+r} v_S v_T$$

and the proof is complete. □

We pause to reformulate this result and to consider some special cases. Recall that if $T \subset \mathbf{m}$ then $v_T \in C(V)$ is unitary. As a consequence, the identity just established can be reformulated as saying that if S and T are subsets of $\mathbf{m}$ then

$$v_T v_S v_T^* = (-1)^{|S|\,|T|+|S\cap T|} v_S.$$

In particular, if $S \subset \mathbf{m}$ and if $j \in \mathbf{m}$ then

$$v_j v_S v_j = (-1)^{|S|+|S\cap j|} v_S.$$

More particularly still, if $|S|$ is even then

$$v_j v_S v_j = \begin{cases} +v_S & (j \notin S) \\ -v_S & (j \in S) \end{cases}$$

whilst if $|S|$ is odd then

$$v_j v_S v_j = \begin{cases} -v_S & (j \notin S) \\ +v_S & (j \in S) \end{cases}.$$

This reformulation and these special cases turn out to be particularly valuable in our analysis of the complex Clifford algebra.

The remaining property of the vectors $\{v_S : S \subset \mathbf{m}\}$ will not be needed in establishing that these vectors form a basis for $C(V)$ but it is important for other reasons and is conveniently disposed of at this point. In order to state the property, we require some notation. For subsets $S \subset \mathbf{m}$ and $T \subset \mathbf{m}$ we denote their *symmetric difference* by $S\Delta T$ as usual, so that

$$S\Delta T = (S - T) \cup (T - S);$$

in addition, we denote by $\varepsilon(S,T)$ the sign $(-1)^k$ where k is the cardinality of the set

$$\{(s,t) : s > t\} \subset S \times T.$$

Theorem 1.1.5 *If $S,T \subset \mathbf{m}$ then $v_S v_T = \varepsilon(S,T) v_{S\Delta T}$.*

Proof Let $S = \{s_1 < \ldots < s_p\}$ and $T = \{t_1 < \ldots < t_q\}$. For $j = 1,\ldots,p$ let k_j denote the cardinality of the set $\{t : s_j > t\} \subset T$ so that $\varepsilon(S,T) = (-1)^k$ where $k = k_1 + \ldots + k_p$. Making repeated use of the Clifford relations,

$$\begin{aligned} v_S v_T &= v_{s_1} \ldots v_{s_p} v_{t_1} \ldots v_{t_q} \\ &= (-1)^{k_1} \ldots (-1)^{k_p} v_{S\Delta T} \\ &= (-1)^k v_{S\Delta T} \\ &= \varepsilon(S,T) v_{S\Delta T} \end{aligned}$$

since $v_j^2 = \mathbf{1}$ for $j \in \mathbf{m}$ and since to arrange the elements of the set $S\Delta T$ in increasing order we must move s_r past each $t \in T$ with $s_r > t$ for $r = p,\ldots,1$ (in that order). □

Actually, the use to which we shall put this result only calls for the weaker result that if $S,T \subset \mathbf{m}$ then

$$v_S v_T = \pm v_{S\Delta T}$$

and does not require a determination of the sign.

We are now able to establish the advertised fact that $\{v_S : S \subset \mathbf{m}\}$ is a basis for $C(V)$. That $\{v_S : S \subset \mathbf{m}\}$ spans $C(V)$ is almost immediate from Theorem 1.1.1, according to which the algebra $C(V)$ is generated by its subspace V; all we need note in addition is that the Clifford relations permit the reduction of any finite product from $\{v_1,\ldots,v_m\}$ to one of the form v_S for some $S \subset \mathbf{m}$. Of course, this already implies that $C(V)$ is finite-dimensional. To see that $\{v_S : S \subset \mathbf{m}\}$ is linearly independent, suppose

$$\sum_{S \subset \mathbf{m}} \mu_S v_S = 0$$

to be a nontrivial relation involving as few nonzero coefficients as possible. This minimality and an application of the idempotent operators $\frac{1}{2}(I+\gamma)$ and $\frac{1}{2}(I-\gamma)$ together show at once that the indices $S \subset \mathbf{m}$ for which $\mu_S \neq 0$ all have the same parity: either $|S|$ is even whenever $\mu_S \neq 0$ or $|S|$ is odd whenever $\mu_S \neq 0$. Now hypothesize that the relation involves (at least) two nonzero coefficients; select j in the symmetric difference of the corresponding pair of indices in $\mathbf{m}$. From the first special case following Theorem 1.1.4 we deduce that

$$\begin{aligned} 0 &= v_j \Big(\sum_{S \subset \mathbf{m}} \mu_S v_S \Big) v_j \\ &= \sum_{S \subset \mathbf{m}} (-1)^{|S|+|S \cap j|} \mu_S v_S \end{aligned}$$

whence

$$0 = \sum_{S \subset \mathbf{m}} (-1)^{|S \cap j|} \mu_S v_S$$

since the sign $(-1)^{|S|}$ is constant over $\{S \subset \mathbf{m} : \mu_S \neq 0\}$. By hypothesis, addition of this relation to the original will result in a nontrivial relation having fewer nonzero coefficients, a patent absurdity. The supposed nontrivial relation cannot have just one nonzero coefficient since v_S is invertible whenever $S \subset \mathbf{m}$. Thus, the supposed nontrivial relation among the vectors $\{v_S : S \subset \mathbf{m}\}$ is nonexistent, so that $\{v_S : S \subset \mathbf{m}\}$ is indeed linearly independent. Of course, it now follows that $C(V)$ has complex dimension $2^{|\mathbf{m}|} = 2^m$.

Theorem 1.1.6 *If $\{v_1, \ldots, v_m\}$ is an orthonormal basis for V then $\{v_S : S \subset \mathbf{m}\}$ is a basis for $C(V)$ so that*

$$\dim_{\mathbb{C}} C(V) = 2^{\dim_{\mathbb{R}} V}.$$

□

A little later, we shall offer an alternative proof that the vectors $\{v_S : S \subset \mathbf{m}\}$ are linearly independent: Theorem 1.1.9 states that $C(V)$ carries a natural positive-definite Hermitian inner product, relative to which $\{v_S : S \subset \mathbf{m}\}$ is an orthonormal basis. Our construction of this natural inner product will be performed with the aid of another natural structure carried by the complex Clifford algebra: namely, a normalized even central linear functional which we call its trace. Our handling of this trace is facilitated by having access to the left regular representation of the complex Clifford algebra. In order not to interrupt the development of the trace, it is convenient to present a brief introduction to the left regular representation at this juncture.