The Padé approximant of a given power series is a rational function of numerator degree L and denominator degree M whose power series agrees with the given one up to degree $L + M$ inclusively. A collection of Padé approximants formed by using a suitable set of values of L and M often provides a means of obtaining information about the function outside its circle of convergence, and of more rapidly evaluating the function within its circle of convergence.

Applications of these ideas in physics, chemistry, electrical engineering, and other areas have led to a large number of generalizations of Padé approximants which are tailor-made for specific applications. Applications to statistical mechanics and critical phenomena are extensively covered, and there are newly extended sections devoted to circuit design, matrix Padé approximation, computational methods, and integral and algebraic approximants.

The book is written with a smooth progression from elementary ideas to some of the frontiers of research in approximation theory. Its main purpose is to make the various techniques described accessible to scientists, engineers, and other researchers who may wish to use them while also presenting the rigorous mathematical theory.

This second edition has been thoroughly updated, with several new sections added, including a substantial new chapter on multiseries approximants.

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

EDITED BY G.-C. ROTA

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Padé Approximants, second edition

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ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

PADÉ APPROXIMANTS
Second Edition

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To our wives
Carroll Thomas *and* **Lucia Graves-Morris**
and to our families
and
to the memory of
Elizabeth Coles Baker

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PREFACE

We are glad that the first edition of these volumes is thought to have achieved its main aim of making mathematical techniques more available, not only to mathematicians, but also to the wider scientific and engineering community.

We have been glad to take the opportunity provided by this edition to incorporate the most salient aspects of the large body of new results which have been obtained since the publication of the original edition. The incorporation of this new material has led to the need to make several significant rearrangements of the previous material.

We wish to record our gratitude for the mathematical contributions and company of Arne Magnus and Helmut Werner, both of them friends who are missed by many of us. The influence of their work is to be found in Chapter 4.

A few infelicities which have been noticed in the original edition have been corrected.

George A. Baker, Jr.
Peter Graves-Morris

PREFACE TO THE FIRST EDITION

These two volumes are intended to serve as a basic text on one approach to the problem of assigning a value to a power series. We have attempted to present the basic results and methods in as transparent a form as possible, in line with the general objectives of the Encyclopedia. The general topic of Padé approximants, which is, among other things, a highly practical method of definition and of construction of the value of a power series, seems to have begun independently at least twice. Padé's claim for credit is based on his thesis (1892), in which he developed the approximants and organized them in a table. He paid particular attention to the exponential function. He was presumably unaware of the prior work of Jacobi (1846), who gave the determinantal representation in his paper on the simplification of Cauchy's solution to the problem of rational interpolation. Also, Padé's work was preceded by that of Frobenius (1881), who derived identities between the neighboring rational fractions of Jacobi. It is interesting to note that Anderson seems to have stumbled upon some Padé approximants for the logarithmic function in 1740. A photograph of H. Padé is to be found in *The Padé Approximant Method and Its Application to Mechanics*, edited by H. Cabannes. A copy of his autographed thesis is to be found in the Cornell University Library.

This work has been distilled from an extensive literature, and *The Essentials of Padé Approximants*, written by one of us, has been an essential reference. We use the abbreviation EPA for this book, and refer to it often for a different or fuller treatment of some of the more advanced topics. While each book is entirely self-contained, our notation is normally compatible with EPA, and to a large extent the books complement each other. An important exception is that the Padé table in EPA is reflected through its main diagonal in our present notation. The proceedings of the Canterbury Summer School and International Conference, edited by the other of us, contain diverse contributions which initiated in print the multidisciplinary view of the subject—a view we hope we have transmitted herein. The many publications which have contributed substantially to our text are listed in the bibliography. We are grateful to our numerous colleagues at Brookhaven, Canterbury, Cornell, Los Alamos, and Saclay in freely discussing so many topics which have made possible the breadth of our treatment. Especially, we thank Roy Chisholm, John Gammel, and Daniel Bessis for many conversations, and the C.E.A. at Saclay, where part of this book was written, for hospitality.

Our hardest task in writing this book was to choose a presentation which is both correct and readily comprehensible. A fully precise system based on rigorous analysis and set-theoretic language would have ensured total obscurity of the more practical techniques. Conversely, omission of all the conditions under which the theorems hold good would be absurdly misleading. We have chosen a level of presentation suitable for the topic in hand. For example, the connectivity of sets is mentioned where it is important, and otherwise it is omitted. The meaning of the order notation is clear in context. Both applications in physics and techniques recently developed are treated in a practical fashion.

Equations are referenced by a default option. Equation (I.6.5.3) is Equation (5.3) of Part I. Chapter 6; the Part and Chapter are dropped by default if they are the same as the source of the reference.

Finally, a spirit of evangelism may be detected in the text. When a review of rational approximation in 1963 can claim that Padé approximants cannot approximate on the entire range $(0, \infty)$ and be believed, a revision of view is overdue.

George A. Baker, Jr.
Peter Graves-Morris
1 October, 1980.