0 Introduction

This book corresponds to a graduate course which has been taught many times at the universities Paris-Sud, Paris-Nord and Paris 7, and other places. The aim of this text is to give the foundation of what is nowadays called microlocal analysis in the C^∞ framework, as it was created in the sixties and seventies by Kohn–Nirenberg, Maslov and Hörmander. Our presentation follows essentially the one given by Hörmander [Hö2]; as for the symplectic geometry, we have been inspired by the lecture notes of Duistermaat [D].

This subject is of growing importance, with a range of applications going beyond the original problems of linear partial differential equations. In particular the link with quantum mechanics is now firmly established, and there is a growing number of books covering more or less specialized parts of the theory. We believe that a short monograph concentrating on the basic principles could be of value, not only for the graduate student, but also for the mathematician who wants to get *quickly* into the subject and to understand its basic mechanisms. For this, the classical PDE framework seemed to us the most suitable one. The basic principles of microlocal analysis are essentially only two: integration by parts and the method of stationary phase. Compared with the article [Hö2] and many of the other presentations, we have insisted even more on the stationary phase method, which appears already in the development of the theory of pseudodifferential operators and also (as in Melin–Sjöstrand [MS]) in the proof of the equivalence of phase functions in the global theory of Fourier integral operators.

Readers of this book are expected to be familiar with the theory of distributions, in particular with the Fourier transform. They should also know some basic functional analysis and differential geometry. The following is a brief outline of the contents of the book :

1) Symbols and oscillatory integrals. Here we introduce some notions related to asymptotic developments and the "non-stationary phase lemma". We also study some properties of Fourier integral operators and distributions.

2) The method of stationary phase. In this chapter we prove the Morse lemma on the normal form of C^∞ functions near a non-degenerate critical point, and apply it in order to find the asymptotic expansion of an integral of the form $\int e^{i\lambda\phi(x)}a(x)dx$, when $\lambda \longrightarrow \infty$ and ϕ has critical points.

3) Pseudodifferential operators. Here we develop the theory of these operators and its symbolic calculus with composition, adjoints and changes of variables.

4) Applications to elliptic operators and L^2 continuity. We construct approximate inverses (parametrices) to elliptic operators, which leads us naturally to leave the class of differential operators. We also study the action of pseudodifferential operators in L^2 and Sobolev spaces.

5) Local symplectic geometry I (Hamilton–Jacobi theory). We recall briefly some basic notions of differential geometry and explain how to solve Hamilton–Jacobi equations locally.

6) The strictly hyperbolic Cauchy problem – construction of a parametrix. We follow the method of P.D. Lax [L], which is a particular case of the general so-called WKB method.

7) The wavefront set (singular spectrum) of a distribution. This is a refinement of the notion of singular support. It not only describes the set where a distribution is singular but also localizes the frequencies that constitute these singularities.

8) Propagation of singularities for operators of real principal type. In the special case of the wave equation this result says that the singular support of a solution is a union of optical rays. The precise formulation of this result requires the notion of wavefront set.

9) Local symplectic geometry II. We discuss canonical transformations and normal forms.

10) Canonical tranformations of pseudodifferential operators. We prove the Egorov theorem, which states that the conjugation of a pseudodifferential operator by a Fourier integral operator gives a new pseudodifferential operator whose principal symbol is obtained by composition by a canonical transformation. This result often permits one to simplify a given pseudodifferential operator.

11) Global theory of Fourier integral operators. We sketch the global theory after establishing its essential ingredient: two non-degenerate phase functions that generate the same Lagrangian manifold give rise to the same classes of oscillatory integrals.

12) Spectral theory for elliptic operators. We prove a theorem of Hörmander on the Weyl asymptotics with a small remainder for self-adjoint elliptic operators of arbitrary order on a compact manifold.

Exercises can be found at the end of each chapter. Some are just for fun and some indicate important results that we did not put in the main text, such as: pseudodifferential operators on manifolds, the sharp Gårding inequality, the Calderón–Vaillancourt theorem (for symbols of type 0,0), 1-dimensional WKB and the Bohr–Sommerfeld quantization condition and some indication on the Weyl quantization of symbols.

We mention other useful monographs: L. Hörmander [Hö4], F. Treves [Tr], M. Taylor [T], M. Shubin [Sh], J. Chazarain and A. Piriou [ChP], V.P. Maslov and M. Fedoriuk [MaFe], V.P. Maslov [Ma1], S. Alinhac and P. Gérard [AGé], D. Robert [R], J-M. Delort [De].

The subject of this book is marked by the influence of L. Hörmander and one of us has had the privilege of being his pupil during some of the most exciting stages of the development of the theory. Many students have followed our lectures through the years and contributed with stimulating questions and remarks. In particular T. Ramond, F. Klopp and L. Nédelec have been helpful with the exercises. From our wifes and children, we have received support and understanding for this job. Many thanks to all these people.

Cambridge University Press
978-0-521-44986-1 - Microlocal Analysis for Differential Operators: An Introduction
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1 Symbols and oscillatory integrals

We shall use the following notation : $\mathbb{R}$ is the set of real numbers, $\mathbb{N} = \{0,1,2,\dots\}$, $\mathbb{R}^n = \mathbb{R}\times\dots\times\mathbb{R}$ (n factors), $\dot{\mathbb{N}} = \mathbb{N}\backslash\{0\}$, $\dot{\mathbb{R}}^n = \mathbb{R}^n\backslash\{0\}$.

An element $\alpha = (\alpha_1,\dots,\alpha_n)$ of $\mathbb{N}^n$ will be called a multi-index and the length of α is the corresponding ℓ^1-norm : $|\alpha| = \alpha_1 + \dots + \alpha_n$. (For points $x\in\mathbb{R}^n$, we denote by $|x|$ or by $\|x\|$ the ordinary Euclidean norm.) We write $x^\alpha = x_1^{\alpha_1}\dots x_n^{\alpha_n}$, $x = (x_1,\dots,x_n)$, $\partial_x^\alpha = \partial_{x_1}^{\alpha_1}\dots\partial_{x_n}^{\alpha_n}$, $\partial_{x_j} = \dfrac{\partial}{\partial x_j}$, $D_x^\alpha = D_{x_1}^{\alpha_1}\dots D_{x_n}^{\alpha_n}$, $D_x = \dfrac{1}{i}\partial_x$, $D_{x_j} = \dfrac{1}{i}\partial_{x_j}$. If $X\subset\mathbb{R}^n$ is open, $k\in\mathbb{N}$, we let $C^k(X)$ denote the (Fréchet) space of k times continuously differentiable functions $X\to\mathbb{C}$. For $k=0$ we get the space $C(X)$ of continuous complex-valued functions and we let $C^\infty(X) = \bigcap_{k\in\mathbb{N}} C^k(X)$ be the (Fréchet) space of infinitely (continuously) differentiable functions. If I is a subset of $\mathbb{R}$, $C^k(X;I)$ is the set of functions $\in C^k(X)$ taking their values in I. Recall that if $u\in C^k(X)$, the support of u, $\operatorname{supp} u$, is by definition the smallest closed subset L of X outside which u vanishes identically. For $k\in\mathbb{N}\cup\{\infty\}$ we let $C_0^k(X) = \{u\in C^k(X)\,;\ \operatorname{supp} u \text{ is compact}\}$. If $M\subset\mathbb{R}^n$ is closed, we let $C^k(M)$ denote the space of restrictions to M of functions in $C^k(\mathbb{R}^n)$. For such functions we define the support as above as a closed subset of M, and we can then define $C_0^k(M)$ as the space of functions in $C^k(M)$ with bounded support.

Let $X\subset\mathbb{R}^n$ be an open set, $0\le\rho\le1$, $0\le\delta\le1$, $m\in\mathbb{R}$, $N\in\mathbb{N}\backslash\{0\}$. Then we have the following

Definition 1.1 $S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ is the space of all $a\in C^\infty(X\times\mathbb{R}^N)$ such that for all compact $K\subset\subset X$ and all $\alpha\in\mathbb{N}^n$, $\beta\in\mathbb{N}^N$, there is a constant $C = C_{K,\alpha,\beta}(a)$ such that

$$|\partial_x^\alpha\,\partial_\theta^\beta\,a(x,\theta)| \le C(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|},\ (x,\theta)\in K\times\mathbb{R}^N.$$

We say that $S^m_{\rho,\delta}$ is the space of symbols of order m and of type (ρ,δ).

It is easy to check that $S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ is a Fréchet (vector) space with the seminorms :

$$P_{K,\alpha,\beta}(a) = \sup_{(x,\theta)\in K\times\mathbb{R}^N}\frac{|\partial_x^\alpha\,\partial_\theta^\beta\,a(x,\theta)|}{(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}}$$

for K compact in X, $\alpha\in\mathbb{N}^n$, $\beta\in\mathbb{N}^N$. (A countable family of seminorms defining the topology is given by the $P_{K_j,\alpha,\beta}$, $j=1,2\dots$, $\alpha\in\mathbb{N}^n$, $\beta\in\mathbb{N}^N$, where $K_1,K_2,\dots$ is an increasing sequence of compact subsets of X such that $X = \bigcup_{j=1}^\infty K_j$.) The operator $\partial_x^\alpha\partial_\theta^\beta$ is continuous from $S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ to $S^{m-|\beta|\rho+|\alpha|\delta}_{\rho,\delta}(X\times\mathbb{R}^N)$.

If $m \le m'$, $\delta \le \delta'$, $\rho \ge \rho'$, then $S^m_{\rho,\delta} \subset S^{m'}_{\rho',\delta'}$. The space of symbols of order $-\infty$ is defined as

$$S^{-\infty}(X \times \mathbb{R}^N) = \Big\{ a \in C^\infty(X \times \mathbb{R}^N) \text{ ; for every compact } K \subset X$$

and all $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$, $M \in \mathbb{R}$, there exists $C = C_{K,\alpha,\beta,M}(a)$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \le C(1+|\theta|)^{-M},\ x \in K,\ \theta \in \mathbb{R}^N \Big\}.$$

This space is also a Fréchet space (with the "best" constants as seminorms), and for every fixed $(\rho,\delta) \in [0,1] \times [0,1]$ we have

$$S^{-\infty}(X \times \mathbb{R}^N) = \bigcap_{m \in \mathbb{R}} S^m_{\rho,\delta}(X \times \mathbb{R}^N).$$

There is no point in introducing spaces $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ with $\rho > 1$ or with $\delta < 0$. For instance if $a \in S^m_{\rho,\delta}$ with $m < 0$ and $\rho > 1$, then applying $|\theta|\,\partial_{|\theta|} = \sum \theta_j \partial_{\theta_j}$ (working in polar coordinates) many times and then integrating, we see that $a \in S^{-\infty}$. Similar phenomena appear for $m \ge 0$, and also for $\delta < 0$.

Example 1.2 Let $a \in C^\infty(X \times \mathbb{R}^N)$ be positively homogeneous of degree m in the region $|\theta| \ge 1$; $a(x,\lambda\theta) = \lambda^m a(x,\theta)$, $\lambda \ge 1$, $|\theta| \ge 1$. Then $a \in S^m_{1,0}(X \times \mathbb{R}^N)$.

Example 1.3 $e^{ix\cdot\xi} \in S^0_{0,1}(\mathbb{R}^n \times \mathbb{R}^n)$. (Here $x\cdot\xi = \sum x_j\,\xi_j$, $x = (x_1,\dots,x_n)$, $\xi = (\xi_1,\dots,\xi_n)$.)

Example 1.4 Let $f \in C^\infty(X \times \mathbb{R}^N\,;\,[0,\infty[)$ (i.e. f is smooth on $X \times \mathbb{R}^N$ and takes its values in $[0,\infty[$), and let f be positively homogeneous of degree 1 for $|\theta| \ge 1$. Then $e^{-f} \in S^0_{\frac12,\frac12}(X \times \mathbb{R}^N)$.

Verification : First, if $g \in C^2(\Omega)$ where $\Omega \subset \mathbb{R}^M$ is open and $g \ge 0$, then for every compact $K \subset \Omega$ we have $|g'(x)| \le C\,g(x)^{1/2}$, $x \in K$, where $g'(x) = (\partial_{x_1} g(x),\dots,\partial_{x_M} g(x))$. In fact, for $x \in K$ and $y \in \mathbb{R}^M$ sufficiently small, we have $0 \le g(x+y) = g(x) + g'(x)\cdot y + C|y|^2$, so that $-g'(x)\cdot y \le g(x) + C|y|^2$. We take $y = -r\,g'(x)$ with $r > 0$ sufficiently small depending on K and on g. Then we get $r\,|g'(x)|^2 \le g(x) + Cr^2|g'(x)|^2$, so if $Cr \le \frac12$ we get the desired estimate with a new constant C. (In general we shall follow the convention that "C" denotes a new constant in every new formula.)

Applying this estimate to f and using the homogeneity, we find

$$|\theta|^{-\frac12}|f'_x(x,\theta)| + |\theta|^{\frac12}|f'_\theta(x,\theta)| \le C_K|f(x,\theta)|^{\frac12},$$

$$(x,\theta) \in K \times \mathbb{R}^N,\ |\theta| \ge 1,\ K \text{ compact } \subset X,$$

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978-0-521-44986-1 - Microlocal Analysis for Differential Operators: An Introduction
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where $f'_x = (\partial_{x_1} f, \dots, \partial_{x_n} f)$, $f'_\theta = (\partial_{\theta_1} f, \dots, \partial_{\theta_N} f)$. Hence (using that $t^k e^{-t} \le C_k$, $t \ge 0$, $k \ge 0$),

$$|f'_x|^k \, |f'_\xi|^\ell \, e^{-f} \le C_{k,\ell,K}(1+|\theta|)^{\frac{k-\ell}{2}}, \quad (x,\theta) \in K \times \mathbb{R}^N,$$

and by induction we find that $\partial_x^\alpha \partial_\theta^\beta e^{-f}$ is a finite sum of terms of the type $a(x,\theta)\,(\partial_x f)^{\tilde\alpha}\,(\partial_\theta f)^{\tilde\beta}\, e^{-f}$ with $\tilde\alpha \in \mathbb{N}^n$, $\tilde\beta \in \mathbb{N}^N$, $|\tilde\alpha| \le |\alpha|$, $|\tilde\beta| \le |\beta|$, $a \in S_{1,0}^{(|\alpha|-|\tilde\alpha|)/2-(|\beta|-|\tilde\beta|)/2}$. From this it follows that $e^{-f} \in S^0_{\frac12,\frac12}$.

Proposition 1.5 *If $a \in S^{m_1}_{\rho,\delta}(X \times \mathbb{R}^N)$, $b \in S^{m_2}_{\rho,\delta}(X \times \mathbb{R}^N)$, then $ab \in S^{m_1+m_2}_{\rho,\delta}(X \times \mathbb{R}^N)$. More generally, the bilinear map*

$$S^{m_1}_{\rho,\delta}(X \times \mathbb{R}^N) \times S^{m_2}_{\rho,\delta}(X \times \mathbb{R}^N) \ni (a,b) \mapsto ab \in S^{m_1+m_2}_{\rho,\delta}(X \times \mathbb{R}^N)$$

is continuous.

Outline of the proof: The first statement is immediate if we express $\partial_x^\alpha \partial_\theta^\beta (ab)$ by means of Leibniz' formula. The only additional work required for the second statement is to recall how to express the continuity of the bilinear map in terms of seminorms (see Exercise 1.5). □

Proposition 1.6 *Let $(a_j)_{j=1}^\infty$ be a bounded sequence in $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ which converges at each point $(x,\theta) \in X \times \mathbb{R}^N$. Then the pointwise limit a belongs to $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and for every $m' > m$ we have $a_j \to a$ in (the topology of) $S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N)$.*

Proof: For $f \in C^2([-\varepsilon,\varepsilon])$ (twice continuously differentiable on $[-\varepsilon,\varepsilon]$), $\varepsilon > 0$ we have :

$$(1.1) \quad |f'(0)| \le C_\varepsilon (\|f\|_{L^\infty}^{\frac12} \|f''\|_{L^\infty}^{\frac12} + \|f\|_{L^\infty}), \quad \|f\|_{L^\infty} = \sup_{[-\varepsilon,\varepsilon]} |f(x)|.$$

(See Exercise 1.6.)

Applying this estimate to the various variables, we obtain by induction that $(a_j)_{j=1}^\infty$ is a Cauchy sequence in $C^k(X \times \mathbb{R}^N)$ for all $k \in \mathbb{N}$ and hence also for $k = \infty$. Consequently, the limit a belongs to $C^\infty(X \times \mathbb{R}^N)$ and $a_j \to a$ in $C^\infty(X \times \mathbb{R}^N)$. It is then clear that $a \in S^m_{\rho,\delta}$.

In order to prove the convergence in $S^{m'}_{\rho,\delta}$, we let K be compact $\subset X$, $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$ and consider for $(x,\theta) \in K \times \mathbb{R}^N$:

$$k_j(x,\theta) = \frac{\partial_x^\alpha \partial_\theta^\beta (a_j - a)}{(1+|\theta|)^{m'-\rho|\beta|+\delta|\alpha|}} = \frac{1}{(1+|\theta|)^{m'-m}} \cdot \frac{\partial_x^\alpha \partial_\theta^\beta (a_j - a)}{(1+|\theta|)^{m-\rho|\beta|+\delta|\alpha|}}.$$

The last factor to the right is uniformly bounded with respect to j, x, θ, hence for every $\varepsilon > 0$, there is an $R_\varepsilon > 0$ such that $|k_j(x,\theta)| < \frac{\varepsilon}{2}$ for

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978-0-521-44986-1 - Microlocal Analysis for Differential Operators: An Introduction
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$x \in K$, $(1+|\theta|) \geq R_\varepsilon$, $j = 1, 2 \ldots$. On the other hand $k_j(x,\theta) \to 0$, $j \to \infty$, uniformly on the compact set given by $x \in K$, $(1+|\theta|) \leq R_\varepsilon$. Hence $k_j(x,\theta) \to 0$ uniformly on $K \times \mathbb{R}^N$. It follows that $a_j - a \to 0$ in $S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N)$. □

Proposition 1.7 *If $m' > m$, then $S^{-\infty}(X \times \mathbb{R}^N)$ is dense in $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ for the topology of $S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N)$.*

Proof: Let $\chi \in C_0^\infty(\mathbb{R}^N)$, $\chi(\theta) = 1$ for $|\theta| \leq 1$, $\chi(\theta) = 0$ for $|\theta| \geq 2$. Then $\chi_j(\theta) = \chi\left(\frac{\theta}{j}\right)$, $j = 1, 2 \ldots$ is a bounded sequence in $S^0_{1,0}(X \times \mathbb{R}^N)$ (the symbols are actually independent of x). In fact, for $\alpha \in \mathbb{N}^N \setminus \{0\}$, $\partial_\theta^\alpha \chi_j(\theta) = j^{-|\alpha|} \chi^{(\alpha)}(\theta/j) = \mathcal{O}((1+|\theta|)^{-|\alpha|})$ uniformly in j, θ, since $j \leq |\theta| \leq 2j$ on the support of $\chi^{(\alpha)}(\theta/j)$. (Here $\chi^{(\alpha)} = \partial_\theta^\alpha \chi$.) If $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ it suffices to put $a_j(x,\theta) = \chi\left(\frac{\theta}{j}\right) a(x,\theta)$ and apply Propositions 1.5, 1.6 as well as the inclusion $S^0_{1,0} \subset S^0_{\rho,\delta}$. □

We next study asymptotic sums of symbols.

Proposition 1.8 *Let $a_j \in S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^N)$, $j = 0, 1, 2 \ldots$ with $m_j \searrow -\infty$, $j \to \infty$. Then there exists $a \in S^{m_0}_{\rho,\delta}(X \times \mathbb{R}^N)$ unique modulo (i.e. up to some element in) $S^{-\infty}(X \times \mathbb{R}^N)$, such that $a - \sum_{0 \leq j < k} a_j \in S^{m_k}_{\rho,\delta}$ for $k = 0, 1, 2, \ldots$.*

Proof: The uniqueness modulo $S^{-\infty}(X \times \mathbb{R}^N)$ follows from : $\bigcap_{k=0}^{\infty} S^{m_k}_{\rho,\delta} = S^{-\infty}$. As for the existence, we may first assume that the sequence (m_j) is strictly decreasing, since we could otherwise regroup the terms with the same value of m_j. For each space $S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^N)$, let $P_{j,0}, P_{j,1}, P_{j,2}, \ldots$ be a sequence of seminorms defining the topology on $S^{m_j}_{\rho,\delta}$. According to Proposition 1.7, for each j we can find $b_j \in S^{-\infty}$ such that $P_{\nu,\mu}(a_j - b_j) \leq 2^{-j}$ for $0 \leq \nu$, $\mu \leq j-1$. Then $\sum_{j \geq k}(a_j - b_j)$ converges in $S^{m_k}_{\rho,\delta}$ for each k, and if we put $a = \sum_0^\infty (a_j - b_j) \in S^{m_0}_{\rho,\delta}$ then

$$a - \sum_{j<k} a_j = -\sum_{j<k} b_j + \sum_k^\infty (a_j - b_j) \in S^{m_k}_{\rho,\delta}.$$

□

This proof uses the standard Cantor diagonalization procedure and is close to the Borel theorem which states that for any family $a_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{N}^n$, there exists $f \in C^\infty(\mathbb{R}^n)$ with $f^{(\alpha)}(0) = \alpha!\, a_\alpha$, $\alpha \in \mathbb{N}^n$, so that the (formal) Taylor–Maclaurin series of f at 0 is $\sum_{\alpha\in\mathbb{N}^n} a_\alpha x^\alpha$. See Exercise 1.7.

If a and a_j have the properties of the proposition, we write $a \sim \sum_j^\infty a_j$ and we call a an (and sometimes "the") asymptotic sum of the a_j. In practice we often need the following result :

Proposition 1.9 *Let $a_j \in S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^N)$, $j = 0, 1, 2\ldots$, with $m_j \searrow -\infty$ and let $a \in C^\infty(X \times \mathbb{R}^N)$ have the properties*

1) For all compact $K \subset X$ and $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^N$, there exists $C_{\alpha,\beta,K}$, $M_{\alpha,\beta} > 0$ such that

$$|\partial_x^\alpha \partial_\theta^\beta a(x,\theta)| \le C_{\alpha,\beta,K}(1 + |\theta|)^{M_{\alpha,\beta}}, \quad (x,\theta) \in K \times \mathbb{R}^N.$$

2) There exists a sequence $m'_k \to -\infty$, $k \to \infty$ such that for all compact $K \subset X$ and $k \in \mathbb{N}$, there exists $C_{K,k} > 0$ such that

$$|a(x,\theta) - \sum_0^{k-1} a_j(x,\theta)| \le C_{K,k}(1 + |\theta|)^{m'_k}, \quad (x,\theta) \in K \times \mathbb{R}^N.$$

Then $a \sim \sum_0^\infty a_j$.

Proof: Let $a' \in S^{m_0}_{\rho,\delta}$ be an asymptotic sum, $a' \sim \sum_0^\infty a_j$. Then $b = a - a'$ has the property 1), and according to 2) we know that for every compact $K \subset X$ and $M \in \mathbb{N}$, there exists $C_{K,M} > 0$ such that $|b(x,\theta)| \le C_{K,M}(1 + |\theta|)^{-M}$ on $K \times \mathbb{R}^N$.

Using (1.1) we find for every sufficiently small $\varepsilon > 0$ and for $(\alpha, \beta) \in \mathbb{N}^{n+N}$, $|(\alpha,\beta)| = 1$:

$$|\partial_x^\alpha \partial_\theta^\beta b(x,\theta)| \le C_\varepsilon\Big((\sup_B |b|)^{\frac12} (\sup_B |(\partial_x^\alpha \partial_\theta^\beta)^2 b|)^{\frac12} + \sup_B |b|\Big),$$

where $B = B((x,\theta),\varepsilon)$ denotes the open Euclidean ball in $\mathbb{R}^{n+N}$ of center (x,θ) and of radius ε. We conclude that

$$|\partial_x^\alpha \partial_\theta^\beta b(x,\theta)| \le \tilde{C}_{K,M}(1 + |\theta|)^{-M} \quad \text{on} \quad K \times \mathbb{R}^N,$$

for every compact $K \subset X$ and every $M \in \mathbb{N}$. Iterating this argument we get $b \in S^{-\infty}$ and hence $a \sim \sum_0^\infty a_j$. □

From now on, we assume that $0 < \rho \le 1$, $0 \le \delta < 1$.

Definition 1.10 A function $\varphi = \varphi(x,\theta) \in C^\infty(X \times \dot{\mathbb{R}}^N)$ is called a phase function if for all $(x,\theta) \in X \times \dot{\mathbb{R}}^N$:

1) $\operatorname{Im}\varphi(x,\theta) \geq 0$,

2) $\varphi(x,\lambda\theta) = \lambda\varphi(x,\theta)$ for all $\lambda > 0$,

3) $d\varphi \neq 0$.

Here $d\varphi = \sum_1^n \frac{\partial\varphi}{\partial x_j} dx_j + \sum_1^N \frac{\partial\varphi}{\partial\theta_k} d\theta_k$ so 3) means that at every point some derivative $\frac{\partial\varphi}{\partial x_j}$ or $\frac{\partial\varphi}{\partial\theta_k}$ is non-vanishing.

If φ is a phase function and $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$, $m + k < -N$, $k \in \mathbb{N}$ then

$$(1.2) \qquad I(a,\varphi)(x) \overset{\text{def}}{=} \int e^{i\varphi(x,\theta)} a(x,\theta)\, d\theta \in C^k(X),$$

and the map $S^m_{\rho,\delta} \ni a \mapsto I(a,\varphi) \in C^k(X)$ is continuous.

We let $\mathcal{D}'(X)$ be the space of (Schwartz) distributions on X (it is the dual space of $\mathcal{D}(X) = C_0^\infty(X)$,) and $\mathcal{D}'^{(k)}(X)$ be the subspace of distributions of order $\leq k$ (it is the dual space of $C_0^k(X)$). The natural duality between distributions and test functions will be expressed by $\langle , \rangle$ and sometimes we write formally $\langle u, \varphi\rangle = \int u(x)\varphi(x)\, dx$, $u \in \mathcal{D}'(X)$, $\varphi \in C_0^\infty(X)$. We shall only use the weak topology on $\mathcal{D}'(X)$ and on $\mathcal{E}'(X) = \{u \in \mathcal{D}'(X); \operatorname{supp} u$ is compact$\}$.

Theorem 1.11 *Let $\varphi(x,\theta)$ be a phase function on $X \times \dot{\mathbb{R}}^N$ and let $0 < \rho \leq 1$, $0 \leq \delta < 1$. Then there is a unique way of defining $I(a,\varphi) \in \mathcal{D}'(X)$ for $a \in S^\infty_{\rho,\delta} \overset{\text{def}}{=} \bigcup_{m\in\mathbb{R}} S^m_{\rho,\delta}$ such that $I(a,\varphi)$ is defined by (1.2) when $a \in S^m_{\rho,\delta}$, $m < -N$ and such that for every $m \in \mathbb{R}$, the map $S^m_{\rho,\delta} \ni a \mapsto I(a,\varphi) \in \mathcal{D}'(X)$ is continuous.*

Moreover, if $k \in \mathbb{N}$ and $m - k\min(\rho, 1-\delta) < -N$, then $S^m_{\rho,\delta} \ni a \mapsto I(a,\varphi) \in \mathcal{D}'^{(k)}(X)$ is continuous.

Proof: The uniqueness is obvious since $S^{-\infty}$ is dense in $S^m_{\rho,\delta}$ for the topology of $S^{m'}_{\rho,\delta}$ if $m' > m$. For the existence, we shall define $I(a,\varphi)$ with the help of formal integrations by parts, using the following result :

Lemma 1.12 *There exist $a_j \in S^0_{1,0}(X \times \mathbb{R}^N)$, $b_j, c \in S^{-1}_{1,0}(X \times \mathbb{R}^N)$ such that the differential operator*

$$L = \sum a_j(x,\theta)\frac{\partial}{\partial\theta_j} + \sum b_j(x,\theta)\frac{\partial}{\partial x_j} + c(x,\theta)$$

satisfies : ${}^tL(e^{i\varphi}) = e^{i\varphi}$. *Here* ${}^tL = -\sum \frac{\partial}{\partial\theta_j} \circ a_j - \sum \frac{\partial}{\partial x_j} \circ b_j + c$ *is the real transpose of* L *satisfying* $\int\int L(f)\, g\, dx\, d\theta = \int\int f\, {}^tL(g)\, dx\, d\theta,\ f, g \in C_0^\infty(X \times \mathbb{R}^N)$.

Proof of the lemma: The function

$$\Phi(x,\theta) = \sum_1^n \overline{\frac{\partial\varphi}{\partial x_j}} \frac{\partial\varphi}{\partial x_j} + |\theta|^2 \sum_1^N \overline{\frac{\partial\varphi}{\partial\theta_j}} \frac{\partial\varphi}{\partial\theta_j}$$

is $\neq 0$ for $|\theta| \neq 0$ and positively homogeneous of degree 2 : $\Phi(x, \lambda\theta) = \lambda^2\Phi(x,\theta)$, $\theta \neq 0$, $\lambda > 0$. Let $\chi(\theta) \in C_0^\infty(\mathbb{R}^N)$ be equal to 1 in a neighborhood of 0 and put

$$\begin{aligned} {}^tL &= \frac{1-\chi(\theta)}{i\,\Phi(x,\theta)} \left(\sum_1^N |\theta|^2 \overline{\frac{\partial\varphi}{\partial\theta_j}} \frac{\partial}{\partial\theta_j} + \sum_1^n \overline{\frac{\partial\varphi}{\partial x_j}} \frac{\partial}{\partial x_j} \right) + \chi(\theta) \\ &= \sum a'_j \frac{\partial}{\partial\theta_j} + \sum b'_j \frac{\partial}{\partial x_j} + c' \end{aligned}$$

with $a'_j \in S^0_{1,0}$, $b'_j \in S^{-1}_{1,0}$, $c' \in S^{-\infty}$. Then ${}^tL(e^{i\varphi}) = e^{i\varphi}$, and $L = {}^t({}^tL)$ will have an expression as in the lemma. □

For $u \in C_0^\infty(X)$, $a \in S^{-\infty}(X \times \mathbb{R}^N)$, we have

$$\text{(1.3)}\quad \begin{aligned} \langle I(a,\varphi), u\rangle &= \iint e^{i\varphi(x,\theta)} a(x,\theta) u(x) dx\, d\theta \\ &= \iint ({}^tL)^k (e^{i\varphi}) a u\, dx\, d\theta = \iint e^{i\varphi(x,\theta)} L^k(a(x,\theta)u(x))\, dx\, d\theta. \end{aligned}$$

If $a \in S^m_{\rho,\delta}$, $u \in C_0^\infty(X)$, then $L^k(au) \in S^{m-kt}_{\rho,\delta}$, where we have put $t = \min(\rho, 1-\delta)$. More precisely, we have the continuous map

$$\text{(1.4)}\quad S^m_{\rho,\delta}(X \times \mathbb{R}^N) \times C_0^\infty(X) \ni (a,u) \mapsto L^k(au) \in S^{m-kt}_{\rho,\delta}(X \times \mathbb{R}^N).$$

For every compact $K \subset \Omega$,

$$\sup_{K\times\mathbb{R}^N} |L^k(au)|\, (1+|\theta|)^{-m+kt} \leq f_{k,K}(a) \sum_{|\alpha|\leq k} \sup_K |\partial^\alpha u(x)|,$$

where $f_k(a)$ is a suitable seminorm on $S^m_{\rho,\delta}$.

For $a \in S^m_{\rho,\delta}$ we choose $k \in \mathbb{N}$ with $m - kt < -N$ and we put

$$\text{(1.5)}\quad \langle I_k(a,\varphi), u\rangle = \int\int e^{i\varphi} L^k(au)\, dx\, d\theta.$$

Then (1.4) shows that $I_k(a,\varphi) \in \mathcal{D}'(X)$ and the estimate after (1.4) shows that $I_k(a,\varphi) \in \mathcal{D}'^{(k)}$ and that the map $S^m_{\rho,\delta} \ni a \mapsto I_k(a,\varphi) \in \mathcal{D}'^{(k)}(X)$ is continuous.