# Restricted colorings of graphs

Noga Alon *
Department of Mathematics
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University, Tel Aviv, Israel
and Bellcore, Morristown, NJ 07960, USA

**Abstract**

The problem of properly coloring the vertices (or edges) of a graph using for each vertex (or edge) a color from a prescribed list of permissible colors, received a considerable amount of attention. Here we describe the techniques applied in the study of this subject, which combine combinatorial, algebraic and probabilistic methods, and discuss several intriguing conjectures and open problems. This is mainly a survey of recent and less recent results in the area, but it contains several new results as well.

*Research supported in part by a United States Israel BSF Grant

# 1 Introduction

Graph coloring is arguably the most popular subject in graph theory. An interesting variant of the classical problem of coloring properly the vertices of a graph with the minimum possible number of colors arises when one imposes some restrictions on the colors available for every vertex. This variant received a considerable amount of attention that led to several fascinating conjectures and results, and its study combines interesting combinatorial techniques with powerful algebraic and probabilistic ideas. The subject, initiated independently by Vizing [51] and by Erdős, Rubin and Taylor [24], is usually known as the study of the *choosability* properties of a graph. In the present paper we survey some of the known recent and less recent results in this topic, focusing on the techniques involved and mentioning some of the related intriguing open problems. This is mostly a survey article, but it contains various new results as well.

A *vertex coloring* of a graph $G$ is an assignment of a color to each vertex of $G$. The coloring is *proper* if adjacent vertices receive distinct colors. The *chromatic number* $\chi(G)$ of $G$ is the minimum number of colors used in a proper vertex coloring of $G$. An *edge coloring* of $G$ is, similarly, an assignment of a color to each edge of $G$. It is *proper* if adjacent edges receive distinct colors. The minimum number of colors in a proper edge-coloring of $G$ is the *chromatic index* $\chi'(G)$ of $G$. This is clearly equal to the chromatic number of the line graph of $G$.

If $G = (V, E)$ is a (finite, directed or undirected) graph, and $f$ is a function that assigns to each vertex $v$ of $G$ a positive integer $f(v)$, we say that $G$ is *$f$-choosable* if, for every assignment of sets of integers $S(v) \subset Z$ to all the vertices $v \in V$, where $|S(v)| = f(v)$ for all $v$, there is a proper vertex coloring $c : V \mapsto Z$ so that $c(v) \in S(v)$ for all $v \in V$. The graph $G$ is *$k$-choosable* if it is $f$-choosable for the constant function $f(v) \equiv k$. The *choice number* of $G$, denoted $ch(G)$, is the minimum integer $k$ so that $G$ is $k$-choosable. Obviously, this number is at least the classical chromatic number $\chi(G)$ of $G$. The choice number of the line graph of $G$, which we denote here by $ch'(G)$, is usually called the *list chromatic index* of $G$, and it is clearly at least the chromatic index $\chi'(G)$ of $G$.

As observed by various researchers ([51], [24], [1]), there are many graphs $G$ for which the choice number $ch(G)$ is strictly larger than the chromatic number $\chi(G)$. A simple example demonstrating this fact is the complete bipartite graph $K_{3,3}$. If $\{u_1, u_2, u_3\}$ and $\{v_1, v_2, v_3\}$ are its two vertex-classes and $S(u_i) = S(v_i) = \{1, 2, 3\} \setminus \{i\}$, then there is no proper vertex coloring assigning to each vertex $w$ a color from its class $S(w)$. Therefore, the choice number of this graph exceeds its chromatic number. In fact, it is easy to show that, for any $k \geq 2$, there are bipartite graphs whose choice number exceeds $k$; a general construction is given in the following section. The gap between the two parameters $ch(G)$ and $\chi(G)$ can thus be arbitrarily large. In view of this, the following conjecture, suggested independently by various researchers including Vizing, Albertson, Collins, Tucker and Gupta, which apparently appeared first in print in the paper of Bollobás and Harris ([15]), is somewhat surprising.

**Conjecture 1.1 (The list coloring conjecture)** *For every graph $G$, $ch'(G) = \chi'(G)$.*

This conjecture asserts that for *line graphs* there is no gap at all between the choice number and the chromatic number. Many of the most interesting results in the area are proofs of special cases of this conjecture, some of which are described in Sections 2, 3 and 4. The proof for the general case (if true) seems extremely difficult, and even some very special cases that have received a considerable amount of attention are still open.

The problem of determining the choice number of a given graph is difficult, even for small graphs with a simple structure. To see this, you may try to convince yourself that the complete bipartite graph $K_{5,8}$ is 3-choosable; a (lengthy) proof appears in [42]. More formally, it is shown in [24] that the problem of deciding if a given graph $G = (V, E)$ is $f$-choosable, for a given function $f : V \mapsto \{2, 3\}$, is $\Pi_2^p$-Complete. Therefore, if the complexity classes $NP$ and $coNP$ differ, as is commonly believed, this problem is stricly harder than the problem of deciding if a given graph is $k$-colorable, which is, of course, $NP$-Complete. (See [27] for the definitions of the complexity classes above.) More results on the complexity of several variants of the choosability problem appear in [38], where it is also briefly shown how some of these variants arise naturally in the study of various scheduling problems.

The study of choice numbers combines combinatorial ideas with algebraic and probabilistic tools. In the following sections we discuss these methods and present the main results and open questions in the area. The paper is organized as follows. After describing, in Section 2, some basic and initial results, we discuss, in Section 3, an algebraic approach and some of its recent consequences. Various applications of probabilistic methods to choosability are considered in Section 4. A new result is obtained in Section 5 which presents a proof of the fact that the choice number of any simple graph with average degree $d$ is at least $\Omega(\log d/\log\log d)$. Thus, the choice number of a simple graph must grow with its average degree, unlike the chromatic number. The final Section 6 contains some concluding remarks and open problems, in addition to those mentioned in the previous sections.

## 2 Some basic results

One of the basic results in graph coloring is Brooks' theorem [17], that asserts that the chromatic number of every connected graph, which is not a complete graph or an odd cycle, does not exceed its maximum degree. The choosability version of this result has been proved, independently, by Vizing [51] (in a slightly weaker form) and by Erdős, Rubin and Taylor [24]. (See also [40]).

**Theorem 2.1** ( [51], [24]) *The choice number of any connected graph $G$, which is not complete or an odd cycle, does not exceed its maximum degree.*

Note that this suffices to prove the validity of the list coloring conjecture for simple graphs of maximum degree 3 whose chromatic index is not 3 (known as *class* 2 graphs), since by Vizing's theorem [50], the chromatic index of such graphs must be 4.

A graph is called *$d$-degenerate* if any subgraph of it contains a vertex of degree at most $d$. By a simple inductive argument, one can prove the following result.

**Proposition 2.2** *The choice number of any $d$-degenerate graph is at most $d+1$.*

This simple fact implies, for example, that every planar graph is 6-choosable. It is not known if every planar graph is 5-choosable; this is conjectured to be the case in [24]- in fact, it may even be true that every planar graph is 4-choosable.

A characterization of all 2-choosable graphs is given in [24]. If $G$ is a connected graph, the *core* of $G$ is the graph obtained from $G$ by repeatedly deleting vertices of degree 1 until there is no such vertex.

**Theorem 2.3 ([24])** *A simple graph is* 2*-choosable if and only if the core of each of its connected components is either a single vertex, or an even cycle, or a graph consisting of two vertices with three even internally disjoint paths between them, where the length of at least two of the paths is exactly* 2.

Of course, one cannot hope for such a simple characterization of the class of all 3-choosable graphs, since, as observed by Gutner it follows easily from the complexity result mentioned in the introduction that the problem of deciding if a given graph is 3-choosable is $NP$-hard; in fact, as shown in [28], this problem is even $\Pi_2^p$-complete.

In Section 1 we saw an example of a graph with choice number that exceeds its chromatic number. Here is an obvious generalization of this construction. Let $H = (U, W)$ be a $k$-uniform hypergraph which is not 2-colorable; that is, every edge $w \in W$ has precisely $k$ elements and for every 2-vertex coloring of $H$ there is a monochromatic edge. If $|W| = n$ we claim that the complete bipartite graph $K_{n,n}$ is not $k$-choosable. Indeed, denote the vertices of $H$ by $1, 2, \ldots$, and let $A = \{a_w : w \in W\}$ and $B = \{b_w : w \in W\}$ be the two vertex classes of $K_{n,n}$. For each $w \in W$, define $S(a_w) = S(b_w) = \{u \in U : u \in w\}$. One can easily check that there is a proper coloring $c$ of the complete bipartite graph on $A \cup B$ assigning to each vertex $a_w$ and $b_w$ a color from its class $S(a_w)$ $(= S(b_w))$ if and only if the hypergraph $H$ is 2-colorable. Thus, by the choice of $H$, the choice number of $K_{n,n}$ is stricly bigger than $k$.

As shown by Erdős [23], for large values of $k$ there are $k$-uniform hypergraphs with at most $n = (1+o(1))\frac{e \ln 2}{4} k^2 2^k$ edges which are not 2-colorable, showing that there are bipartite graphs with that many vertices on each side whose choice number exceeds $k$. We note that this estimate is nearly sharp, as a very simple probabilistic argument shows that, if $n < 2^{k-1}$, then $K_{n,n}$ is $k$-choosable. Indeed, given a list $S(v)$ of $k$ colors for each vertex $v$ in the two vertex classes $A$ and $B$, let $S$ be the set of all the colors used in the union of all the lists and let us choose a random partition $(S_A, S_B)$ of $S$ into two disjoint parts, where, for each $s \in S$ randomly and independently, $s$ is chosen to be in $S_A$ or in $S_B$ with equal probability. The colors in $S_A$ will be used to color vertices in $A$ and those in $S_B$ to color the vertices in $S_B$. For a fixed vertex $a$ in $A$, the probability that its coloring will fail-that is, we will not be able to color it by a color from $S_A$- is precisely $1/2^k$, as this is the probability that all the colors in its class $S(a)$ were chosen to be in $S_B$. A similar estimate holds for the members of $B$, and hence the probability that there exists a vertex that will fail to receive a color is at most $|A \cup B|/2^k < 1$. This estimate can be slightly improved, using the method (or the result) of Beck ([10]), but the above simple argument suffices to demonstrate the relevance of probabilistic techniques in the study of choice numbers.

The *total chromatic number* of a graph $G$, denoted by $\chi''(G)$, is the minimum number of colors required to color all the vertices and edges of $G$, so that adjacent or incident elements receive distinct colors. The following conjecture is due to Behzad [11].

**Conjecture 2.4 (The total coloring conjecture)** *The total chromatic number of every simple graph $G$ with maximum degree $\Delta$ is at most $\Delta + 2$.*

There are several papers dealing with this conjecture, and the following estimates are known. If $G$ is a simple graph on $n$ vertices with maximum degree $\Delta$ then, as shown by Hind [33], [34]:

$$\chi''(G) \leq \Delta + 1 + 2\lceil\sqrt{\Delta}\rceil,$$

and

$$\chi''(G) \leq \Delta + 1 + 2\lceil\frac{n}{\Delta}\rceil.$$

Chetwynd and Häggkvist [31] showed that, if $t! > n$, then

$$\chi''(G) \leq \Delta + 1 + t.$$

Note that, as observed by the authors of [15], the validity of Conjecture 1.1 (the list coloring conjecture) would imply that, for every simple graph with maximum degree $\Delta$,

$$\chi''(G) \leq \chi'(G) + 2 \leq \Delta + 3.$$

Indeed, let $S = \{1, 2, \ldots, \chi'(G) + 2\}$ be our set of colors. Start with an arbitrary proper vertex coloring of $G$ using these colors; this certainly exists, for example, by Brooks' Theorem and by the fact that $\chi'(G) \geq \Delta$. Now associate with each edge $e$ of $G$ a list $S(e)$ of all the colors in $S$ except the ones appearing on its two ends. By the list coloring conjecture, there is a proper edge coloring of $G$ using, for each edge $e$, a color from $S(e)$; this would give a proper total coloring of $G$. The fact that $|S| \leq \Delta + 3$ now follows from Vizing's theorem ([50]). It seems, however, that getting a $\Delta + O(1)$ upper estimate for the total chromatic number of a simple graph with maximum degree $\Delta$ should be much easier than getting a similar bound for the list chromatic index of such a graph.

## 3 An algebraic approach and its applications

An algebraic technique that, in various cases, supplies useful information on the choice numbers of given graphs, has been developed by M. Tarsi and the present author in [9]. In this section we describe this method and present some of its recent applications.

A subdigraph $H$ of a directed graph $D$ is called *Eulerian* if the indegree $d_H^-(v)$ of every vertex $v$ of $H$ is equal to its outdegree $d_H^+(v)$. Note that we do not assume that $H$ is connected. $H$ is *even* if it has an even number of edges, otherwise, it is *odd*. Let $EE(D)$ and $EO(D)$ denote the numbers of even and odd Eulerian subgraphs of $D$, respectively. (For convenience we agree that the empty subgraph is an even Eulerian subgraph.) The following result is proved in [9].

**Theorem 3.1** *Let $D = (V, E)$ be a digraph, and define $f : V \mapsto Z$ by $f(v) = d_D^+(v) + 1$, where $d_D^+(v)$ is the outdegree of $v$. If $EE(D) \neq EO(D)$, then $D$ is $f$-choosable.*

Note that the assertion of the theorem for the special case of acyclic digraphs, which implies Proposition 2.2, can be proved by a simple inductive argument. The general case seems much more difficult. To prove this theorem, we need the following simple statement.

**Lemma 3.2** *Let $P = P(x_1, x_2, \dots, x_n)$ be a polynomial in $n$ variables over the ring of integers $Z$. Suppose that the degree of $P$ as a polynomial in $x_i$ is at most $d_i$ for $1 \le i \le n$, and let $S_i \subset Z$ be a set of $d_i + 1$ distinct integers. If $P(x_1, x_2, \dots, x_n) = 0$ for all $n$-tuples $(x_1, \dots, x_n) \in S_1 \times S_2 \times \dots \times S_n$, then $P \equiv 0$.*

**Proof** We apply induction on $n$. For $n = 1$, the lemma is simply the assertion that a non-zero polynomial of degree $d_1$ in one variable can have at most $d_1$ distinct zeros. Assuming that the lemma holds for $n-1$, we prove it for $n$ $(n \ge 2)$. Given a polynomial $P = P(x_1, \dots, x_n)$ and sets $S_i$ satisfying the hypotheses of the lemma, let us write $P$ as a polynomial in $x_n$- that is,

$$P = \sum_{i=0}^{d_n} P_i(x_1, \dots, x_{n-1}) x_n^i,$$

where each $P_i$ is a polynomial with $x_j$-degree bounded by $d_j$. For each fixed $(n-1)$-tuple $(x_1, \dots, x_{n-1}) \in S_1 \times S_2 \times \dots \times S_{n-1}$, the polynomial in $x_n$ obtained from $P$ by substituting the values of $x_1, \dots, x_{n-1}$ vanishes for all $x_n \in S_n$, and is thus identically 0. Thus $P_i(x_1, \dots, x_{n-1}) = 0$ for all $(x_1, \dots, x_{n-1}) \in S_1 \times \dots \times S_{n-1}$. Hence, by the induction hypothesis, $P_i \equiv 0$ for all $i$, implying that $P \equiv 0$. This completes the induction and the proof of the lemma. □

The *graph polynomial* $f_G = f_G(x_1, x_2, \dots, x_n)$ of a directed or undirected graph $G = (V, E)$ on a set $V = \{v_1, \dots, v_n\}$ of $n$ vertices is defined by $f_G(x_1, x_2, \dots, x_n) = \Pi\{(x_i - x_j) : i < j \ , \ \{v_i, v_j\} \in E\}$. This polynomial has been studied by various researchers, starting already with Petersen [44] in 1891. See also, for example, [46], [39].

For $1 \le i \le n$, let $S_i \subset Z$ be a set of $d_i + 1$ distinct integers. For each $i$, $1 \le i \le n$, let $Q_i(x_i)$ be the polynomial $Q_i(x_i) = \Pi_{s \in S_i}(x_i - s)$. Let $\mathcal{I}$ be the ideal generated by the polynomials $Q_i$ in the ring of polynomials $Z[x_1, \dots, x_n]$. It is obvious that if $f_G(x_1, \dots, x_n) \in \mathcal{I}$, then $f_G$ vanishes on every common zero of all the polynomials $Q_i$. But this means that $f_G$

vanishes on every $(x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n$; hence, for each assignment of values $x_i \in S_i$, there is an edge $v_i v_j$ of $G$ with $x_i = x_j$. Therefore, there is no proper vertex coloring of $G$ assigning to each vertex $v_i$ a color from its set $S_i$. The following Nullstellensatz-type result asserts that the converse is also true.

**Proposition 3.3** *Let $G = (V, E)$ be a graph on the set of vertices $V = \{v_1, \ldots, v_n\}$, and let $S_i$, $1 \le i \le n$, be sets of integers. Let $f_G = f_G(x_1, \ldots, x_n)$ be the graph polynomial of $G$, and let $Q_i(x_i)$ and $\mathcal{I}$ be as above. Then $f_G \in \mathcal{I}$ if and only if there is no proper vertex coloring $c$ of $G$ satisfying $c(v_i) \in S_i$, for all $1 \le i \le n$.*

**Proof** We have already seen that, if there is a coloring as above, then $f_G$ is not in $\mathcal{I}$. It remains to show that if there is no such coloring, then $f_G \in \mathcal{I}$. The assumption that the required coloring does not exist is equivalent to the statement:

$$f_G(x_1, \ldots, x_n) = 0 \quad \text{for every } n\text{-tuple} \quad (x_1, \ldots, x_n) \in S_1 \times S_2 \times \ldots \times S_n. \tag{1}$$

For each $i$, $1 \le i \le n$, put

$$Q_i(x_i) = \Pi_{s \in S_i}(x_i - s) = x_i^{d_i+1} - \sum_{j=0}^{d_i} q_{ij} x_i^j.$$

Observe that,

$$\text{if } x_i \in S_i \text{ then } Q_i(x_i) = 0\text{- that is, } x_i^{d_i+1} = \textstyle\sum_{j=0}^{d_i} q_{ij} x_i^j. \tag{2}$$

Let $\overline{f_G}$ be the polynomial obtained by writing $f_G$ as a linear combination of monomials and replacing, repeatedly, each occurrence of $x_i^{f_i}$ $(1 \le i \le n)$, where $f_i > d_i$, by a linear combination of smaller powers of $x_i$, using the relations (2). The resulting polynomial $\overline{f_G}$ is clearly of degree at most $d_i$ in $x_i$, for each $1 \le i \le n$, and satisfies $\overline{f_G} \equiv f_G$ $(mod\ \mathcal{I})$. Moreover, $\overline{f_G}(x_1, \ldots, x_n) = f_G(x_1, \ldots, x_n)$, for all $(x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$, since the relations (2) hold for these values of $x_1, \ldots, x_n$. Therefore, by (1), $\overline{f_G}(x_1, \ldots, x_n) = 0$ for every $n$-tuple $(x_1, \ldots, x_n) \in S_1 \times \ldots \times S_n$ and hence, by Lemma 3.2, $\overline{f_G} \equiv 0$. This implies that $f_G \in \mathcal{I}$, and completes the proof. □

The special case of the last proposition, for the case in which all the sets $S_i$ are equal, implies that, for every fixed polynomial $Q(x)$ of one variable with $k$ distinct integer roots, a graph $G$ is not $k$-colorable if and only if the graph polynomial $f_G$ lies in the ideal generated by the polynomials $Q(x_i)$. In fact, the assumption that the roots of $Q(x)$ are integral is not essential, as the proof works equally well in the ring of polynomials $K[x_1, \ldots, x_n]$ over any field $K$. See [9] for more details. This result is related to a theorem of Kleitman and Lovász ([41]) , who applied a method similar to that of [39], and showed that a graph $G = (V, E)$ is not $k$-colorable if and only if $f_G$ lies in the ideal generated by the set of all graph polynomials of complete graphs on $k+1$ vertices among those in $V$. As shown by De Loera in [19], the set of graph polynomials of complete $(k+1)$-graphs, as well as the set of polynomials $Q(x_i)$ above, are both universal Gröbner bases for the ideals they generate. See [19] for more details.

It is not too difficult to see that the coefficients of the monomials that appear in the standard representation of $f_G$ as a linear combination of monomials can be expressed in terms of the orientations of $G$. For each oriented edge $e = (v_i, v_j)$ of $G$, define its *weight* $w(e)$ by $w(e) = x_i$ if $i < j$, and $w(e) = -x_i$ if $i > j$. The weight $w(D)$ of an orientation $D$ of $G$ is defined to be the product $\Pi w(e)$, where $e$ ranges over all oriented edges $e$ of $D$. Clearly $f_G = \sum w(D)$, where $D$ ranges over all orientations of $G$. This is simply because each term in the expansion of the product $f_G = \Pi\{(x_i - x_j) : i < j \ , \ \{v_i, v_j\} \in E\}$ corresponds to a choice of the orientation of the edge $\{v_i, v_j\}$ for each edge $\{v_i, v_j\}$ of $G$. Let us call an oriented edge $(v_i, v_j)$ of $G$ *decreasing* if $i > j$. An orientation $D$ of $G$ is called *even* if it has an even number of decreasing edges; otherwise, it is called *odd*.

For non-negative integers $d_1, d_2, \ldots, d_n$, let $DE(d_1, \ldots, d_n)$ and $DO(d_1, \ldots, d_n)$ denote, respectively, the sets of all even and odd orientations of $G$ in which the outdegree of the vertex $v_i$ is $d_i$, for $1 \le i \le n$. By the last paragraph, the following lemma holds.

**Lemma 3.4** *In the above notation*

$$f_G(x_1, \ldots, x_n) = \sum_{d_1, \ldots, d_n \ge 0} (|DE(d_1, \ldots, d_n)| - |DO(d_1, \ldots, d_n)|) \Pi_{i=1}^{n} x_i^{d_i} \ . \quad \square$$