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INTRODUCTION

Discrete mathematics has had many practical applications in recent years and this is only one of the reasons for its increasing dynamism. The study of finite structures is a broad area which has a unity not merely of description but also in practice, since many of the structures studied give results which can be applied to other, apparently dissimilar structures. Apart from the applications, which themselves generate problems, internally there are still many difficult and interesting problems in finite geometry and combinatorics, and we are happy to be able to demonstrate progress.

It was a great pleasure to see several Russian colleagues participating both because they were able to do so, some for the first time, and because this is an area of Mathematics not as diffuse in Russia as elsewhere. It was also good to see the participation of a significant number of talented, younger colleagues, but at the same time sad to note the difficulty they are having in finding permanent positions.

The conference papers are here divided into themes. The division is somewhat artificial as some papers could be placed in more than one group. The style of mathematics is very much resolving problems rather than the construction of grand theories. There are still many puzzling features about the sub-structures of finite projective spaces, as well as about finite strongly regular graphs, finite projective planes, and other particular finite diagram geometries. Finite groups are as ever a strong theme for several reasons. There is still much work to be done to give a clear geometric identification of the finite simple groups. There are also many problems in characterizing structures which either have a particular group acting on them or which have some degree of symmetry from a group action.

Generalized polygons

Bader and *Lunardon* together and *Lunardon* alone give new constructions of classical hexagons. *Buekenhout* and *Van Maldeghem* show that there are no surprises in the action of a classical group on a hexagon or octagon. *De Smet* and *Van Maldeghem* give a new geometrical characterization of the finite classical hexagons and characterise some finite Moufang hexagons, and *van Bon* determines some extended generalized hexagons having certain geometric and group-theoretical properties. *Payne* constructs a coherent configuration inside a translation generalized quadrangle. *Brouwer* shows that the subgeometry of a polygon induced on the objects in general position with respect to a given flag is connected.

Graphs and their groups

Brouwer, *Fon-der-Flaass*, and *Shpectorov* find the three graphs which locally are the incidence graph of the unique biplane on 7 points. *Haemers* shows that a strongly regular graph on 76 points with consistent parametric conditions cannot exist. *Munemasa*, *Pasechnik*, and *Shpectorov* characterize the graphs of alternating and quadratic forms over $GF(2)$, whereas *Munemasa* and *Shpectorov* characterize alternating forms over all larger fields. In a similar vein, *Pasechnik* describes the graph of a certain $GF(3)$-geometry, which leads to a characterization of Fischer's sporadic simple groups. *Soicher* shows that the Lyons simple group has no distance-transitive representation and hence determines all faithful multiplicity-free representations of this group.

Finite Desarguesian planes

A spread of a Hermitian curve in a Desarguesian plane is a set of non-tangent lines partitioning the points of the curve. *Baker*, *Ebert*, *Korchmáros*, and *Szőnyi* study such spreads with the property that no line of the spread contains the pole of any other line in the spread and deduce a result on the linear code of the plane. A minimal blocking set in $\mathrm{PG}(2,q)$ has size b satisfying $q+\sqrt{q}+1 \leq b \leq q\sqrt{q}+1$, with the bounds being achieved for square q by a Baer subplane and a Hermitian curve. Related to this, *Blokhuis* and *Metsch* study strong representative systems and show that, for $q \geq 25$, one cannot have $b = q\sqrt{q}$. A nucleus of a set of $q+1$ points in $\mathrm{PG}(2,q)$ is a point not in the set such that every line through it meets the set; it is known that the number of nuclei is at most $q-1$. *Blokhuis* and *Mazzocca* use a mapping to $\mathrm{PG}(3,q)$ to deduce more about the structure of sets in $\mathrm{PG}(2,q)$ with the maximum number of nuclei. *Glynn* relates the code of $\mathrm{PG}(2,q)$ with q even to the study of nonics, where a nonic is defined to be either a conic plus its nucleus or a line pair, real or imaginary, less the point of intersection. *Gordon* obtains a formula for the number of projectively distinct k-arcs in $\mathrm{PG}(n,q)$ and also finds an efficient algorithm for determining this number; this is applied to finding the number of projectively distinct k-arcs and projectively distinct complete k-arcs in $\mathrm{PG}(2,11)$ and $\mathrm{PG}(2,13)$. *Hirschfeld* and *Voloch* obtain further results on the characterization of sets of points in $\mathrm{PG}(2,q)$ with at most three points on a line as cubic curves.

Higher-dimensional projective spaces

Storme and *Szőnyi* examine k-arcs having many points in common with a normal rational curve. For odd q, it is elementary that a plane k-arc not contained in a conic has at most $(q+3)/2$ points on the conic. The problem is much more difficult in higher dimensions; it is shown that, for large q and bounded n, if an arc has more than $(q+1)/2$ points in common with a normal rational curve, it is contained in the curve. A partial flock of a quadric cone in $\mathrm{PG}(3,q)$ is a set of disjoint conics on the cone and a flock is a set of q conics forming a partition of the cone less its vertex. *Thas*, *Herssens* and *De Clerck* survey the known flocks and construct a new flock for $q = 11$ which generalizes to a partial flock of size 11 for any $q \equiv -1 \pmod{12}$. An ovoid of a quadric is a set of points meeting every generator (subspace of maximum dimension lying on the quadric) precisely once. *Moorhouse* describes a 9-dimensional lattice which defines simply an ovoid on a hyperbolic quadric in $\mathrm{PG}(7,p)$, p prime. These ovoids had previously been constructed by the author from the E_8 root lattice.

Non-Desarguesian planes

Ho investigates Singer groups S acting on a projective plane of order n and proves, among other things, that, if the multiplier group $M(S)$ has even order, each subgroup of S is invariant under the involution of $M(S)$, except possibly if $n = 16$ and S is non-abelian. *Jha* and *Wene* find the number of central units of a commutative semifield plane. *Johnson* characterises certain translation planes of order q^2 as equivalent to particular subsets of cardinality q of the collineation group $\mathrm{P\Gamma L}(2,q)$. *Wettl* extends results on nuclei in a Desarguesian plane to the non-Desarguesian case.

Block designs

A $t-(v,k,\lambda)$ design is an incidence structure of v points and b blocks with k points on each block and the essential property that through t points there are precisely λ blocks. Such structures exist for all t by Teirlinck's theorem, but the known examples mostly have very large λ. *Cameron* and *Praeger* investigate block-transitive designs with $5 \leq t \leq 8$. Among other results they determine the possible automorphism groups of block-transitive 6-designs and flag-transitive 5-designs.

Polar spaces

Buekenhout presents an old result of Parmentier which axiomatizes a pair (P,π), where P is a projective space and π a polarity, so that the axioms defining P alone are weaker than usual. *Shult* develops the axiomatization of Veldkamp spaces for point-line geometries.

Diagram geometries

Buekenhout and *King* study flag-transitive diagrams of rank 3 such that the residues of 0-elements are dual Petersen graphs, of 1-elements are generalized digons, and of 2-elements are finite linear spaces. The linear space turns out to be either a projective plane, in which case there are precisely two geometries, or the complete graph on four vertices. In the latter case there are no geometries for which the group acts primitively on the 2-elements, but examples are given of the imprimitive case.

A grid is an incidence structure of points and lines such that any line has at least two points and any two points are incident with at most one line, with the additional property that the lines fall into two classes such that two lines intersect if and only if they belong to different classes; these intersections give all the points. *Meixner* and *Pasini* describe all known extensions of grids. *Ghinelli* classifies the flag-transitive rank 3 geometries in which the planes are linear spaces with constant line size and the point residues are classical generalized quadrangles other than grids. *Huybrechts* gives a new proof of the commutativity of the division ring for a thick, residually-connected D_n-geometry.

A generalized Fischer space is a partial linear space in which any two intersecting lines generate a subspace that is an affine plane or its dual. Subject only to some non-degeneracy conditions, *Cuypers* gives a complete classification of these spaces. *Mühlherr* describes a geometric method of constructing, from the diagrams, Coxeter groups as subgroups of other Coxeter groups.

Generalized hexagons and BLT-sets

L. Bader *G. Lunardon**

Abstract

An alternative construction for the dual $G_2(q)$-hexagon is given for q odd and different from 3^n.

1. Introduction

In [4], W.M. Kantor has constructed the generalized quadrangle associated with the Fisher-Thas-Walker flock as a group coset geometry starting from the dual $G_2(q)$-hexagon. Analyzing Kantor's construction, the following question arises in a natural way: is it possible to define new points and new lines in a generalized quadrangle Q associated with a flock of the quadratic cone, in such a way that the new point-line geometry H is a generalized hexagon?

For q odd, we prove that the only possibility is that Q is the Kantor generalized quadrangle constructed in [4] and H is the dual $G_2(q)$-hexagon. If $q \neq 3^n$, using a twisted cubic of $PG(3,q)$ we obtain an alternative construction of the dual $G_2(q)$-hexagon similar to the construction of a generalized quadrangle using a BLT-set ([6] or [11]). For q even, we are able to prove a strong connection between the existence of H and the $(q+1)$-arcs of $PG(3,q)$ but the answer is not complete due to difficulties of the same type that arise when studying BLT-sets in even characteristic.

We would like to express our thanks to S. E. Payne, J. A. Thas and H. Van Maldeghem for critical remarks on earlier versions of this paper, and to W. M. Kantor for useful discussions during his visit in Rome. In particular, Theorem 2.1 generalizes a result of W. M. Kantor (private communication).

2. Generalized hexagons as group coset geometries

Let $s,t > 1$ be natural numbers. We denote by $\hat{F}$ a set of $s+1$ elements. Let G be a group of order s^2t^3. For any u in $\hat{F}$, fix the subgroups $A_1(u)$, $A_2(u)$, $A_3(u)$, $A_4(u)$ of G such that $A_1(u) \leq A_2(u) \leq A_3(u) \leq A_4(u)$, where $|A_1(u)| = t$, $|A_2(u)| = st$, $|A_3(u)| = st^2$ and $|A_4(u)| = s^2t^2$. Define a point-line geometry $H = (P, L, I)$ as follows:

$$P = \{\mathcal{I}, A_4(u)g, A_2(u)g, g : g \in G, u \in \hat{F}\}$$

*. The authors are members of G.N.S.A.G.A. of C.N.R. and have partial financial support by Italian M.U.R.S.T.

$$L = \{[u], A_3(u)g, A_1(u)g : g \in G, u \in \hat{F}\}$$

where $\mathcal{I}$ and $[u]$ are symbols and the incidences are $\mathcal{I}I[u]$, $A_4(u)I[u]$ and $gIA_1(u)g$ for all $g \in G$ and $u \in \hat{F}$, while $A_i(u)gIA_{i+1}(v)h$ if and only if $u = v$ and $g \in A_{i+1}(v)h$ with $i = 1, 2, 3$.

Theorem 2.1 *$H = (P, L, I)$ is a generalized hexagon with parameters (s, t) if and only if, for all distinct i, j, h, m, n in $\{1, 2..., s+1\}$, the following conditions hold:*
1) $A_4(i) \cap A_1(j) = 1$,
2) $A_3(i) \cap A_1(j)A_1(h) = 1$,
3) $A_3(i) \cap A_2(j) = 1$,
4) $A_2(i)A_2(j) \cap A_1(h) = 1$,
5) $A_2(i) \cap A_1(j)A_1(h)A_1(m) = 1$,
6) $A_2(i) \cap A_1(j)A_1(i)A_1(h) = A_1(i)$,
7) $A_1(i) \cap A_1(j)A_1(h)A_1(m)A_1(n) = 1$,
8) $A_1(i) \cap A_1(j)A_1(h)A_1(m)A_1(h) = 1$,
9) $A_1(i)A_1(j) \cap A_1(j)A_1(i) = A_1(i) \cup A_1(j)$,
10) $A_1(i) \cap A_1(j)A_1(h)A_1(j)A_1(h) = 1$.

Proof. With a direct calculation, we can prove that if H is a generalized hexagon, then the ten conditions are satisfied.

Conversely, if we suppose that the conditions 1-10 hold, then no circuit of length less than 12 can exist in H. The point-line geometry H has exactly $(1+t)(1+st+s^2t^2)$ points and exactly $(s+1)(1+st+s^2t^2)$ lines. Each line is incident with exactly $s+1$ points, and each point is incident with exactly $t+1$ lines. Moreover, H contains at least one circuit of length 12, while no circuit of length less than 12 can exist. Consequently H is a generalized hexagon by [12] p. 5. □

By condition 9 of Theorem 2.1, if H is a generalized hexagon then G cannot be abelian.

Starting from H, we define a new point-line geometry $Q(H)$ in the following way. The points of $Q(H)$ are $\mathcal{I}$, $A_3(u)g$ and g, with $g \in G$ and $u \in \hat{F}$. The lines of $Q(H)$ are $[u]$ and $A_2(u)g$, with $g \in G$ and $u \in \hat{F}$. The line $[u]$ is incident with $\mathcal{I}$ and $A_3(u)g$ for all g in G, while all other incidences are given by inclusion.

Corollary 2.2 *Let H be a generalized hexagon, whose parameters s and t are equal. Then $Q(H)$ is a generalized quadrangle with parameters (s^2, s) if and only if $A_2(u)A_2(v) \cap A_2(w) = 1$ for all distinct $u, v, w \in \hat{F}$.*

Proof. If $s = t$, the corollary it is an easy consequence of Section 10.1 of [10]. □

The hypothesis $s = t$ is required in Corollary 2.2, because the point $\mathcal{I}$ is incident with $s+1$ lines, while the point $A_3(u)g$ is incident with $t+1$ lines. If $Q(H)$ is a generalized quadrangle, then the conditions of Theorem 2.1 can be simplified (see [1]). In [5] W.M. Kantor has given an explicit description of the subgroups $A_1(u)$, $A_2(u)$, $A_3(u)$ and $A_4(u)$ for the known generalized hexagons. In [4] W.M. Kantor has proved that if H is the dual $G_2(q)$-hexagon then $Q(H)$ is a generalized quadrangle when $q \equiv -1 \pmod 3$. Moreover, $Q(H)$ is the generalized quadrangle associated with the Fisher-Thas-Walker flock of the quadratic cone (see [13]).

3. $(q+1)$-arcs

Let $F = GF(q)$ and $\hat{F} = F \cup \{\infty\}$. In the following, we always denote by G the group whose elements are those of $F^2 \times F \times F^2$, and whose product is defined by

$$(\alpha, c, \beta)(\alpha', c', \beta') = (\alpha + \alpha', c + c' + \alpha' \cdot \beta, \beta + \beta')$$

where $\alpha, \beta, \alpha', \beta' \in F^2$, $c, c' \in F$, and $\alpha' \cdot \beta = \alpha' \beta^T$. The center of G is the set $Z = \{(0, c, 0) : c \in F\}$ and the group $\bar{G} = G/Z$ is elementary abelian. Moreover, we can regard $\bar{G}$ as a four dimensional vector space over F. For each element g of G, let g^* be the preimage in G of the 1-space of $\bar{G}$ spanned by $\bar{g} = gZ$. If $\bar{g}, \bar{h}$ are elements of $\bar{G}$, we notice that $(\bar{g}, \bar{h}) = [g, h]$ defines a non singular alternating F-bilinear form on $\bar{G}$; if q is even, $\bar{g} \mapsto g^2$ defines a quadratic form associated with $(\ ,\ )$. Thus, $\bar{G}$ is equipped with a symplectic or orthogonal geometry.

If $[g, h] = 1$, then $[g^*, h^*] = 1$. Thus maximal elementary abelian subgroups of G are preimages of totally isotropic (or singular) 2-spaces of $\bar{G}$. Therefore a maximal elementary abelian subgroup of G has order q^3.

Let $PG(3, q)$ be the three dimensional projective space associated with the F-vector space $\bar{G}$. For each $u \in \hat{F}$ if $A_3(u) = A_2(u)Z$, then $A_2(u)$ is elementary abelian because it is canonically isomorphic to the subgroup $A_3(u)/Z$ of $\bar{G}$. So $A_3(u)$ is a maximal elementary abelian subgroup of G. Thus, $L_u = A_3(u)/Z$ is a totally isotropic (or singular) line of the projective space $PG(3, q)$.

Theorem 3.1 *Suppose that H is a generalized hexagon, $A_3(u) = A_2(u)Z$. Denote by p_u, L_u, α_u respectively the point $A_1(u)Z/Z$* [1]*, the line $A_3(u)/Z$ and*

1. As Z is contained in $A_4(v)$, $A_1(u) \cap Z = 1$. Therefore $A_1(u)Z/Z$ is a 1-dimensional vector subspace of $\bar{G}$.

the plane $A_4(u)/Z$ of $PG(3,q)$. The set $\Sigma = \{p_u : u \in \hat{F}\}$ is a (q+1)-arc of PG(3,q). Moreover, L_u is a tangent line of Σ at p_u and α_u is the osculating plane of Σ at p_u.

Proof. Let u and v be two distinct elements of $\hat{F}$. By property 9 of Theorem 2.1, if $g_u \in A_1(u)$ and $g_v \in A_1(v)$ then $[g_u, g_v] = (\bar{g}_u, \bar{g}_v) = 1$ if and only if $g_u = 1$ or $g_v = 1$. Thus, $p_u^\perp \cap \Sigma = \{p_u\}$ where $p_u^\perp$ is the polar plane of p_u with respect to the polarity of $PG(3,q)$ defined by the F-bilinear form (,) of $\bar{G}$.

If $p_v \in \alpha_u$, then $A_1(v) \leq A_1(v)Z \leq A_4(u)$. By property 1 of Theorem 2.1, this implies $u = v$. Therefore, if u and v are different elements of $\hat{F}$, the point p_v does not belong to α_u. Thus, $\alpha_u \cap \Sigma = \{p_u\}$.

Let u, v, w be elements of $\hat{F}$. If $v \neq w$, then the plane $< p_v, L_w >$ of $PG(3,q)$ is defined by a subgroup of G of order q^4 containing $A_1(v)$ and $A_3(w)$. By property 1 of Theorem 2.1, this subgroup is $A_1(v)A_3(w)$. If p_u belongs to the plane $< p_v, L_w >$ then $A_1(u) \leq A_1(u)Z \leq A_1(v)A_3(w)$. By property 2 of Theorem 2.1, either $u = v$ or $u = w$. Thus, $< p_v, L_w > \cap \Sigma = \{p_v, p_w\}$. We have proved that each plane through L_w contains at most one point of Σ different from p_w. As Σ contains exactly q points different from p_w, there are q planes containing the line L_w and a point of Σ different from p_w. So there is exactly one plane through L_w, which contains only one point of Σ. This implies $p_w^\perp = \alpha_w$. Moreover, three distinct points of Σ are never collinear.

By way of contradiction, we suppose that four points p_u, p_v, p_w, p_x of Σ are coplanar. Let $\bar{K}$ be the 3-dimensional subspace of $\bar{G}$, which defines the plane $\beta =< p_u, p_v, p_w >$. The subgroup K of G, which is the preimage of $\bar{K}$, has order q^4 and contains the subgroups $A_1(u), A_1(v), A_1(w), A_1(x)$ because the points p_u, p_v, p_w, p_x belong to the plane β.

Let $g_u, h_u \in A_1(u)$, $g_v, h_v \in A_1(v)$, $g_w, h_w \in A_1(w)$ and suppose that $g_u g_v g_w = h_u h_v h_w \xi$ where $\xi \in Z$. We have

$$\begin{aligned} & g_u g_v g_w h_w^{-1} h_v^{-1} h_u^{-1} \\ & \quad = g_u h_u^{-1} g_v h_v^{-1} g_w h_w^{-1} [h_v g_v^{-1}, h_u][h_w g_w^{-1}, h_v][h_w g_w^{-1}, h_u] = \xi. \end{aligned}$$

Therefore, $(g_u h_u^{-1})(g_v h_v^{-1})(g_w h_w^{-1}) \in Z$ because G' is contained in Z. Hence $(g_u h_u^{-1})(g_v h_v^{-1})(g_w h_w^{-1})$ defines the 0-vector of $\bar{G}$. Thus, the vectors $g_u h_u^{-1} Z$, $g_v h_v^{-1} Z$, $g_w h_w^{-1} Z$ of $\bar{G}$ are linear dependent. As the points p_u, p_v and p_w are never collinear, we have $g_u h_u^{-1} \in Z$ or $g_v h_v^{-1} \in Z$ or $g_w h_w^{-1} \in Z$. This implies $g_u = h_u$, $g_v = h_v$ and $g_w = h_w$ because $Z \leq A_4(x)$ and $A_4(x) \cap A_1(y) = 1$ for all distinct x and y in $\hat{F}$. Then $A_1(u)A_1(v)A_1(w) \cap h_u h_v h_w Z = h_u h_v h_w$. Moreover, for $h_u h_v h_w = 1$, we have also proved $A_1(u)A_1(v)A_1(w) \cap Z = 1$.

With the same argument, we can prove that if $g_u g_v g_w = h_u h_v h_w$, then $g_u = h_u$, $g_v = h_v$ and $g_w = h_w$. Hence, the subset $A_1(u)A_1(v)A_1(w)$ of G has order q^3. As $A_1(u)$, $A_1(v)$, $A_1(w)$ and Z are subgroups of K, we have

$K = A_1(u)A_1(v)A_1(w)Z$.

The subset $A_1(u)A_1(v)A_1(w)A_1(v)$ is contained in K because $A_1(u)$, $A_1(v)$ and $A_1(w)$ are contained in K. Let $g_u \in A_1(u)$, $g_v, h_v \in A_1(v)$ and $g_w \in A_1(w)$ be elements of G different from 1. Then the element $g_u g_v g_w h_v = g_u g_v h_v g_w [g_w^{-1}, h_v^{-1}]$ does not belong to $A_1(u)A_1(v)A_1(w)$ because $[g_w^{-1}, h_v^{-1}] \neq 1$. Thus, the subset $A_1(u)A_1(v)A_1(w)A_1(v)$ of K has order $> q^3$. As $A_1(x) \cap A_1(u)A_1(v)A_1(w)A_1(v) = 1$ by property 8 of Theorem 2.1, the subset $A_1(x)A_1(u)A_1(v)A_1(w)A_1(v)$ has order $> q^4$; also, it is contained in the subgroup K, because $A_1(x)$ and $A_1(u)A_1(v)A_1(w)A_1(v)$ are subsets of K. We have the required contradiction because K has order q^4.

This completes the proof that Σ is a $(q+1)$-arc of $PG(3,q)$.

For each $u \in \hat{F}$, we denote by β_u the osculating plane of Σ at p_u[2]. Let p_u and p_v be two distinct points of Σ. Define

$$\mathcal{C}^* = \{< p_u, p_w > \cap \beta_v : w \in \hat{F}, w \neq u\}$$

As Σ is a $(q+1)$-arc of $PG(3,q)$, $\mathcal{C}^*$ is a q-arc of the plane β_v. Let $\mathcal{C} = \mathcal{C}^* \cup \{L_u \cap \beta_v\}$.

By way of contradiction, we suppose that a line N of β_v incident with the point $L_u \cap \beta_v$ contains three points of $\mathcal{C}$. Then the plane $< N, p_u >$ is incident with three points of Σ. As the line L_u is contained in $< N, p_u >$, this is impossible. Therefore, each line of β_v contains at most two points of $\mathcal{C}$. This implies that $\mathcal{C}$ is a $(q+1)$-arc of β_v. By the definition of tangent of a $(q+1)$-arc of $PG(3,q)$, we have proved that the line L_u is tangent of Σ at p_u.

If q is odd, L_u is the tangent line of Σ at p_u. If q is even, L_u is one of the two tangents of Σ in p_u. In both cases, L_u is contained in β_u. As the osculating plane contains exactly one point of Σ, we have $\alpha_u = \beta_u$ because α_u is the unique plane containing L_u such that $\alpha_u \cap \Sigma = \{p_u\}$. □

Corollary 3.2 *If H is a generalized hexagon with $A_3(u) = A_2(u)Z$, and $q = p^n$ with p a prime number, then $p \neq 3$.*

Proof. Suppose $p = 3$. Thus, Σ is a twisted cubic and all the osculating planes of Σ contain a fixed line L ([8] Section 43). If u, v, w are mutually distinct elements of $\hat{F}$, then $< p_u, p_v, p_w >^{\perp} = \alpha_u \cap \alpha_v \cap \alpha_w = L$. Therefore, the F-bilinear form (,) has non-trivial radical. As this is impossible, we have a contradiction. □

In [4] W.M. Kantor has proved that if H is isomorphic to the dual $G_2(q)$-

2. For q even, the osculating plane at p_u is the plane defined by the two tangents of Σ at p_u; for q odd, Σ is a normal rational curve and β_u is the osculating plane of the normal rational curve at p_u (i.e. the plane through p_u with intersection multiplicity 3). For more details see [8]

hexagon, then Σ is a twisted cubic; also, there is a canonical way to construct the dual $G_2(q)$-hexagon starting from a twisted cubic (see [4] Remark 2). Thus, we have proved the following corollary

Corollary 3.3 *Let q be odd. If H be a generalized hexagon with $A_3(u) = A_2(u)Z$, then H is isomorphic to the dual $G_2(q)$-hexagon.*

4. The case q odd

In this section we always suppose that q is odd.

Let $W(5,q)$ be the polar space associated with a symplectic polarity of $PG(5,q)$. Embed the symplectic 3-space $W(3,q)$ in $W(5,q)$ and let p be a fixed point of $W(5,q)$ such that $p \notin W(3,q) \subset p^\perp \subset W(5,q)$, where "$\perp$" is relative to $W(5,q)$. We can introduce coordinates in $PG(5,q)$ in such a way that $W(5,q)$ is the polar space associated with the alternating bilinear form

$$b((x_0,\alpha,\beta,x_5),(y_0,\gamma,\delta,y_5)) = x_0y_5 - x_5y_0 + \alpha\cdot\delta - \beta\cdot\gamma$$

where $\alpha,\beta,\gamma,\delta \in F^2$, $\alpha\cdot\delta = \alpha\delta^T$ and $\beta\cdot\gamma = \beta\gamma^T$. If $p = (0,0,0,0,0,1)$, then $p^\perp$ has equation $x_0 = 0$. We can embed $W(3,q)$ in $W(5,q)$ in such a way that $W(3,q)$ is the 3-dimensional subspace of $W(5,q)$ with equations $x_0 = x_5 = 0$.

The nonsingular collineation τ defined by the upper triangular matrix

$$M(a,b,c,d,e) = \begin{pmatrix} 1 & a & b & c & d & e \\ 0 & 1 & 0 & 0 & 0 & c \\ 0 & 0 & 1 & 0 & 0 & d \\ 0 & 0 & 0 & 1 & 0 & -a \\ 0 & 0 & 0 & 0 & 1 & -b \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

fixes $W(5,q)$ and all the subspaces of $p^\perp$ incident with p. We will identify τ with $M(a,b,c,d,e)$. The group $\tilde{G} = \{M(a,b,c,d,e) : a,b,c,d,e \in F\}$ acts sharply transitive on the points of $PG(5,q) \setminus p^\perp$. Moreover, the map

$$g = (a,b,e,c,d) \mapsto \tilde{g} = M(a,b,c,d,ac+bd-2e)$$

is an isomorphism between G and $\tilde{G}$ (see [11]).

Let $o = (1,0,0,0,0,0)$. Then $W(3,q)$ lies in the 3-dimensional subspace $S = p^\perp \cap o^\perp$ of $PG(5,q)$. Let $\Sigma = \{r_u : u \in \hat{F}\}$ be the twisted cubic of S such that $r_u^\perp \cap p^\perp \cap o^\perp$ is the osculating plane of Σ at r_u for each $u \in \hat{F}$. Let L_u be the tangent line of Σ at r_u $(u \in \hat{F})$. We define a point-line geometry $H(\Sigma)$ in the following way: