LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Managing Editor: Professor J.W.S. Cassels, Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England

The books in the series listed below are available from booksellers, or, in case of difficulty, from Cambridge University Press.

34 Representation theory of Lie groups, M.F. ATIYAH *et al*
36 Homological group theory, C.T.C. WALL (ed)
39 Affine sets and affine groups, D.G. NORTHCOTT
46 p-adic analysis: a short course on recent work, N. KOBLITZ
49 Finite geometries and designs, P. CAMERON, J.W.P. HIRSCHFELD & D.R. HUGHES (eds)
50 Commutator calculus and groups of homotopy classes, H.J. BAUES
57 Techniques of geometric topology, R.A. FENN
59 Applicable differential geometry, M. CRAMPIN & F.A.E. PIRANI
66 Several complex variables and complex manifolds II, M.J. FIELD
69 Representation theory, I.M. GELFAND *et al*
74 Symmetric designs: an algebraic approach, E.S. LANDER
76 Spectral theory of linear differential operators and comparison algebras, H.O. CORDES
77 Isolated singular points on complete intersections, E.J.N. LOOIJENGA
79 Probability, statistics and analysis, J.F.C. KINGMAN & G.E.H. REUTER (eds)
80 Introduction to the representation theory of compact and locally compact groups, A. ROBERT
81 Skew fields, P.K. DRAXL
82 Surveys in combinatorics, E.K. LLOYD (ed)
83 Homogeneous structures on Riemannian manifolds, F. TRICERRI & L. VANHECKE
86 Topological topics, I.M. JAMES (ed)
87 Surveys in set theory, A.R.D. MATHIAS (ed)
88 FPF ring theory, C. FAITH & S. PAGE
89 An F-space sampler, N.J. KALTON, N.T. PECK & J.W. ROBERTS
90 Polytopes and symmetry, S.A. ROBERTSON
91 Classgroups of group rings, M.J. TAYLOR
92 Representation of rings over skew fields, A.H. SCHOFIELD
93 Aspects of topology, I.M. JAMES & E.H. KRONHEIMER (eds)
94 Representations of general linear groups, G.D. JAMES
95 Low-dimensional topology 1982, R.A. FENN (ed)
96 Diophantine equations over function fields, R.C. MASON
97 Varieties of constructive mathematics, D.S. BRIDGES & F. RICHMAN
98 Localization in Noetherian rings, A.V. JATEGAONKAR
99 Methods of differential geometry in algebraic topology, M. KAROUBI & C. LERUSTE
100 Stopping time techniques for analysts and probabilists, L. EGGHE
101 Groups and geometry, ROGER C. LYNDON
103 Surveys in combinatorics 1985, I. ANDERSON (ed)
104 Elliptic structures on 3-manifolds, C.B. THOMAS
105 A local spectral theory for closed operators, I. ERDELYI & WANG SHENGWANG
106 Syzygies, E.G. EVANS & P. GRIFFITH
107 Compactification of Siegel moduli schemes, C-L. CHAI
108 Some topics in graph theory, H.P. YAP
109 Diophantine analysis, J. LOXTON & A. VAN DER POORTEN (eds)
110 An introduction to surreal numbers, H. GONSHOR
111 Analytical and geometric aspects of hyperbolic space, D.B.A. EPSTEIN (ed)
113 Lectures on the asymptotic theory of ideals, D. REES
114 Lectures on Bochner-Riesz means, K.M. DAVIS & Y-C. CHANG
115 An introduction to independence for analysts, H.G. DALES & W.H. WOODIN
116 Representations of algebras, P.J. WEBB (ed)
117 Homotopy theory, E. REES & J.D.S. JONES (eds)
118 Skew linear groups, M. SHIRVANI & B. WEHRFRITZ
119 Triangulated categories in the representation theory of finite-dimensional algebras, D. HAPPEL
121 Proceedings of *Groups - St Andrews 1985*, E. ROBERTSON & C. CAMPBELL (eds)
122 Non-classical continuum mechanics, R.J. KNOPS & A.A. LACEY (eds)
124 Lie groupoids and Lie algebroids in differential geometry, K. MACKENZIE
125 Commutator theory for congruence modular varieties, R. FREESE & R. MCKENZIE
126 Van der Corput's method of exponential sums, S.W. GRAHAM & G. KOLESNIK
127 New directions in dynamical systems, T.J. BEDFORD & J.W. SWIFT (eds)

128 Descriptive set theory and the structure of sets of uniqueness, A.S. KECHRIS & A. LOUVEAU
129 The subgroup structure of the finite classical groups, P.B. KLEIDMAN & M.W.LIEBECK
130 Model theory and modules, M. PREST
131 Algebraic, extremal & metric combinatorics, M-M. DEZA, P. FRANKL & I.G. ROSENBERG (eds)
132 Whitehead groups of finite groups, ROBERT OLIVER
133 Linear algebraic monoids, MOHAN S. PUTCHA
134 Number theory and dynamical systems, M. DODSON & J. VICKERS (eds)
135 Operator algebras and applications, 1, D. EVANS & M. TAKESAKI (eds)
136 Operator algebras and applications, 2, D. EVANS & M. TAKESAKI (eds)
137 Analysis at Urbana, I, E. BERKSON, T. PECK, & J. UHL (eds)
138 Analysis at Urbana, II, E. BERKSON, T. PECK, & J. UHL (eds)
139 Advances in homotopy theory, S. SALAMON, B. STEER & W. SUTHERLAND (eds)
140 Geometric aspects of Banach spaces, E.M. PEINADOR and A. RODES (eds)
141 Surveys in combinatorics 1989, J. SIEMONS (ed)
142 The geometry of jet bundles, D.J. SAUNDERS
143 The ergodic theory of discrete groups, PETER J. NICHOLLS
144 Introduction to uniform spaces, I.M. JAMES
145 Homological questions in local algebra, JAN R. STROOKER
146 Cohen-Macaulay modules over Cohen-Macaulay rings, Y. YOSHINO
147 Continuous and discrete modules, S.H. MOHAMED & B.J. MÜLLER
148 Helices and vector bundles, A.N. RUDAKOV et al
149 Solitons, nonlinear evolution equations and inverse scattering, M.J. ABLOWITZ & P.A. CLARKSON
150 Geometry of low-dimensional manifolds 1, S. DONALDSON & C.B. THOMAS (eds)
151 Geometry of low-dimensional manifolds 2, S. DONALDSON & C.B. THOMAS (eds)
152 Oligomorphic permutation groups, P. CAMERON
153 L-functions and arithmetic, J. COATES & M.J. TAYLOR (eds)
154 Number theory and cryptography, J. LOXTON (ed)
155 Classification theories of polarized varieties, TAKAO FUJITA
156 Twistors in mathematics and physics, T.N. BAILEY & R.J. BASTON (eds)
157 Analytic pro-*p* groups, J.D. DIXON, M.P.F. DU SAUTOY, A. MANN & D. SEGAL
158 Geometry of Banach spaces, P.F.X. MÜLLER & W. SCHACHERMAYER (eds)
159 Groups St Andrews 1989 volume 1, C.M. CAMPBELL & E.F. ROBERTSON (eds)
160 Groups St Andrews 1989 volume 2, C.M. CAMPBELL & E.F. ROBERTSON (eds)
161 Lectures on block theory, BURKHARD KÜLSHAMMER
162 Harmonic analysis and representation theory for groups acting on homogeneous trees, A. FIGA-TALAMANCA & C. NEBBIA
163 Topics in varieties of group representations, S.M. VOVSI
164 Quasi-symmetric designs, M.S. SHRIKANDE & S.S. SANE
165 Groups, combinatorics & geometry, M.W. LIEBECK & J. SAXL (eds)
166 Surveys in combinatorics, 1991, A.D. KEEDWELL (ed)
167 Stochastic analysis, M.T. BARLOW & N.H. BINGHAM (eds)
168 Representations of algebras, H. TACHIKAWA & S. BRENNER (eds)
169 Boolean function complexity, M.S. PATERSON (ed)
170 Manifolds with singularities and the Adams-Novikov spectral sequence, B. BOTVINNIK
172 Algebraic varieties, GEORGE R. KEMPF
173 Discrete groups and geometry, W.J. HARVEY & C. MACLACHLAN (eds)
174 Lectures on mechanics, J.E. MARSDEN
175 Adams memorial symposium on algebraic topology 1, N. RAY & G. WALKER (eds)
176 Adams memorial symposium on algebraic topology 2, N. RAY & G. WALKER (eds)
177 Applications of categories in computer science, M.P. FOURMAN, P.T. JOHNSTONE, & A.M. PITTS (eds)
178 Lower K- and L-theory, A. RANICKI
179 Complex projective geometry, G. ELLINGSRUD, C. PESKINE, G. SACCHIERO & S.A. STRØMME (eds)
180 Lectures on ergodic theory and Pesin theory on compact manifolds, M. POLLICOTT
181 Geometric group theory I, G.A. NIBLO & M.A. ROLLER (eds)
182 Geometric group theory II, G.A. NIBLO & M.A. ROLLER (eds)
183 Shintani zeta functions, A. YUKIE
184 Arithmetical functions, W. SCHWARZ & J. SPILKER
185 Representations of solvable groups, O. MANZ & T.R. WOLF
186 Complexity: knots, colourings and counting, D.J.A. WELSH
187 Surveys in combinatorics, 1993, K. WALKER (ed)
190 Polynomial invariants of finite groups, D.J. BENSON

London Mathematical Society Lecture Note Series. 183

Shintani Zeta Functions

Akihiko Yukie
Oklahoma State University

CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521448048

First published 1993

A catalogue record for this publication is available from the British Library

ISBN 978-0-521-44804-8 paperback

Transferred to digital printing 2008

To my parents Kenzo and Fumiko Yukie

CAMBRIDGE

Cambridge University Press
978-0-521-44804-8 - Shintani Zeta Functions
Akihiko Yukie
Frontmatter
More information

Table of contents

Preface

The content of this book is taken from my manuscripts ‘On the global theory of Shintani zeta functions I–V’ which were originally intended for publication in ordinary journals. However, because of its length and the lack of a book on prehomogeneous vector spaces, it has been suggested to publish them together in book form.

It has been more than 25 years since the theory of prehomogeneous vector spaces began. Much work has been done on both the global theory and the local theory of zeta functions. However, we concentrate on the global theory in this book. I feel that another book should be written on the local theory of zeta functions in the future.

The purpose of this book is to introduce an approach based on geometric invariant theory to the global theory of zeta functions for prehomogeneous vector spaces.

This book consists of four parts. In Part I, we introduce a general formulation based on geometric invariant theory to the global theory of zeta functions for prehomogeneous vector spaces. In Part II, we apply the methods in Part I and determine the principal part of the zeta function for Siegel's case, i.e. the space of quadratic forms. In Part III, we handle relatively easy cases which are required to handle the case in Part IV. In Part IV, we use the results in Parts I–III to determine the principal part of the zeta function for the space of pairs of ternary quadratic forms.

We expanded the introduction of the original manuscripts to help non-experts to have a general idea of the subject. What we try to discuss in the introduction is the history of the subject, and what is required to prove the existence of densities of arithmetic objects we are looking for. Even though the theory of prehomogeneous vector spaces involves many topics, we concentrate on two aspects of the theory, i.e. the global theory and the local theory, in the introduction.

Parts I–III of this book correspond to Parts I–III of the above manuscripts, and Part IV of this book corresponds to Parts IV and V of the above manuscripts. Since the manuscripts were originally intended for publication in ordinary journals, certain changes were made to make this book more comprehensible and self-contained.

However, it is impossible to make this book completely self-contained, and we have to require a reasonable background in adelic language, basic group theory, and geometric invariant theory. For this, we assume that the reader is familiar with the following four books and two papers

[1] A. Borel, Some finiteness properties of adele groups over number fields,
[2] A. Borel, *Linear algebraic groups*,
[28] G. Kempf, Instability in invariant theory,
[35] F. Kirwan, *Cohomology of quotients in symplectic and algebraic geometry*,
[46] D. Mumford and J. Fogarty, *Geometric invariant theory*,
[79] A. Weil, *Basic number theory*.

Weil's book [79] is a standard place to learn basic materials on adelic language. Since we do not depend on class field theory, it is enough for the reader to be familiar with the first several chapters of Weil's book. Borel's paper [1] is a place to learn properties of Siegel domains. We need two facts in geometric invariant theory. One is the Hilbert–Mumford criterion of stability, and the other is the rationality of the equivariant Morse stratification. Mumford–Fogarty [46] and Kirwan [35] are the standard books to learn geometric invariant theory and equivariant Morse theory. The rationality of the equivariant Morse stratification was proved by G. Kempf in his paper [28]. However, even though the proofs of the above two facts are technically involved, the statements of these facts are fairly comprehensible and do not require a special background to understand. Therefore, if the reader is unfamiliar with these subjects, I recommend the reader not to worry about the proofs of the statements in this book which we quote from geometric invariant theory and look at the above documents later if necessary.

We have three original results in this book. One is a generalization of 'Shintani's lemma' to $\mathrm{GL}(n)$ concerning estimates of the smoothed Eisenstein series. Shintani proved this lemma for $\mathrm{GL}(2)$ in [64]. The statement of the result is Theorem (3.4.31). The second result is the determination of the principal part of the zeta function for the space of quadratic forms. The statement of the result is Theorem (4.0.1). Shintani himself studied this case and determined the poles of the associated Dirichlet series for quadratic forms which are positive definite in [65]. The last and the main result of this book is the determination of the principal part of the zeta function for the space of pairs of ternary quadratic forms. The statement of the result is Theorem (13.2.2). We discuss the relevance of these results in the introduction.

D. Wright contributed to this book in many places. He suggested the use of 'Wright's principle' in §3.7 after he read the first manuscript of my paper [86]. Also §0.5 is largely from his note. He also found the reference concerning Omar Khayyam when we wrote our paper [84], and helped me to find some references in this book. I would like to give a hearty thanks to him. As I mentioned above, this book is based on geometric invariant theory. For this, I owe a great deal to D. Mumford for teaching me geometric invariant theory and equivariant Morse theory. I was staying at Institute for Advanced Study during the academic year 1989–1990, and at Sonderforschungsbereich 170 Göttingen during the academic year 1990–1991 while I was writing the manuscript of this book. I would like to thank them for their support of this project. This work was partially supported by NSF Grants DMS-8803085, DMS-9101091.

Akihiko Yukie
February 1992, Stillwater, Oklahoma, USA

Notation

For a finite set A, the cardinality of A is denoted by $\#A$. If f, g are functions on a set X and $|f(x)| \le Cg(x)$ for some constant C independent of $x \in X$, we denote $f(x) \ll g(x)$. If $x, y \in \mathbb{R}$, we also use the classical notation $x \ll y$ if y is a much larger number than x. Since we use this classical notation only for numbers, and not for functions, we hope the meaning of this notation will be clear from the context.

Suppose that G is a locally compact group and Γ is a discrete subgroup of G contained in the maximal unimodular subgroup of G. For any left invariant measure dg on G, we choose a left invariant measure dg (we use the same notation, but the meaning will be clear from the context) on $X = G/\Gamma$ so that

$$\int_G f(g)dg = \int_X \sum_{\gamma \in \Gamma} f(g\gamma)dg.$$

We denote the fields of rational, real, and complex numbers by $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ respectively. We denote the ring of rational integers by $\mathbb{Z}$. The set of positive real numbers is denoted by $\mathbb{R}_+$. For any ring R, $R^\times$ is the set of invertible elements of R. Let k be a number field, and o_k its integer ring. Let $\mathfrak{M}, \mathfrak{M}_\infty, \mathfrak{M}_\mathbb{R}, \mathfrak{M}_\mathbb{C}, \mathfrak{M}_f$ be the set of all the places, all the infinite, real, imaginary, finite places of k respectively. Let $\mathbb{A}_f$ (resp. $\mathbb{A}_f^\times$) be the restricted product of the k_v's (resp. $k_v^\times$'s) over $v \in \mathfrak{M}_f$. Let k_∞ (resp. $k_\infty^\times$) be the product of the k_v's (resp. $k_v^\times$'s) over $v \in \mathfrak{M}_\infty$. Then $\mathbb{A} = k_\infty \times \mathbb{A}_f$, $\mathbb{A}^\times = k_\infty^\times \times \mathbb{A}_f^\times$. If $x \in \mathbb{A}$ or $\mathbb{A}^\times$, we denote the finite (resp. infinite) part of x by x_f (resp. x_∞). If V is a vector space over k, we define $V_\mathbb{A}, V_\infty, V_f$ similarly. Let $\mathscr{S}(V_\mathbb{A}), \mathscr{S}(V_\infty), \mathscr{S}(V_f)$ be the spaces of Schwartz–Bruhat functions.

For any place v, k_v is the completion at v. If $v \in \mathfrak{M}_f$, $o_v \subset k_v$ is, by definition, the integer ring of k_v. Let $|\ |$ be the adelic absolute value. The absolute value of k_v is denoted by $|\ |_v$. For $x \in \mathbb{A}^\times$, we denote the product of the $|x|_v$'s over all $v \in \mathfrak{M}_f$ (resp. $v \in \mathfrak{M}_\infty$) by $|x|_f$ (resp. $|x|_\infty$). For $v \in \mathfrak{M}_f$, let π_v be the prime element, and $|\pi_v|_v = q_v^{-1}$. Note that if v is imaginary and $|x|$ is the usual absolute value, $|x|_v = |x|^2$.

Let r_1, r_2 be the numbers of real and imaginary places respectively. Let h, R, and e be the class number, regulator, and the number of roots of unity of k respectively. Let Δ_k be the discriminant of k. Let $\mathfrak{C}_k = 2^{r_1}(2\pi)^{r_2} hRe^{-1}$. We choose a Haar measure dx on $\mathbb{A}$ so that $\int_{\mathbb{A}/k} dx = 1$. For any finite place v, we choose a Haar measure dx_v on k_v so that $\int_{o_v} dx_v = 1$. We use the ordinary Lebesgue measure dx_v for v real, and $dx_v \wedge d\bar{x}_v$ for v imaginary. Then $dx = |\Delta_k|^{-\frac{1}{2}} \prod_v dx_v$ (see [79, p. 91]).

For $\lambda \in \mathbb{R}_+$, let $\underline{\lambda}$ be the idele whose component at v is $\lambda^{\frac{1}{[k:\mathbb{Q}]}}$ if $v \in \mathfrak{M}_\infty$ and 1 if $v \in \mathfrak{M}_f$. Clearly, $|\underline{\lambda}| = \lambda$. We identify $\mathbb{R}_+$ with a subgroup of $\mathbb{A}^\times$ by the map $\lambda \to \underline{\lambda}$. Let $\mathbb{A}^1 = \{x \in \mathbb{A}^\times \mid |x| = 1\}$. Then $\mathbb{A}^\times \cong \mathbb{A}^1/k^\times \times \mathbb{R}_+$, and $\mathbb{A}^1/k^\times$ is compact. We choose a Haar measure $d^\times t^1$ on $\mathbb{A}^1$ so that $\int_{\mathbb{A}^1/k^\times} d^\times t^1 = 1$. Using this measure, we choose a Haar measure $d^\times t$ on $\mathbb{A}^\times$ so that

$$\int_{\mathbb{A}^\times} f(t)d^\times t = \int_0^\infty \int_{\mathbb{A}^1} f(\underline{\lambda} t^1) d^\times \lambda d^\times t^1,$$

where $d^\times\lambda = \lambda^{-1}d\lambda$. For any finite place v, we choose a Haar measure $d^\times t_v$ on $k_v^\times$ so that $\int_{o_v^\times} d^\times t_v = 1$. Let $d^\times t_v(x) = |x|_v^{-1}dx$ if v is real, and $d^\times t_v(x) = |x|_v^{-1}dx\wedge d\bar{x}$ if v is imaginary. Then $d^\times t = \mathfrak{C}_k^{-1}\prod_v d^\times t_v$ (see [79, p. 95]).

Let $<>= \prod_v <>_v$ be a character of $\mathbb{A}/k$. Let v be a finite place. Suppose that $<>_v$ is trivial on $\pi_v^{-c_v}o_v$ and non-trivial on $\pi_v^{-c_v-1}o_v$. Then we define $a_v = \pi_v^{c_v}$. Let $\mathrm{e}(x) = e^{2\pi\sqrt{-1}x}$. If v is a real place, there exists $a_v \in k_v^\times$ such that $< x >_v = \mathrm{e}(a_v x)$, and if v is an imaginary place, there exists $a_v \in k_v^\times$ such that $< x >_v = \mathrm{e}(a_v x + \overline{a_v x})$. For almost all v, $c_v = 0$. Let $\mathfrak{a} = (a_v)_v \in \mathbb{A}^\times$. Then $|\mathfrak{a}| = |\Delta_k|^{-1}$ (see [79, p. 113]). The idele $\mathfrak{a}$ is called the difference idele of k.

Let $\zeta_k(s)$ be the Dedekind zeta function. As in [79], we define

$$Z_k(s) = |\Delta_k|^{\frac{s}{2}}\left(\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\right)^{r_1}\left((2\pi)^{-s}\Gamma(s)\right)^{r_2}\zeta_k(s).$$

We define $\mathfrak{R}_k = \mathrm{Res}_{s=1} Z_k(s)$.

For a character ω of $\mathbb{A}^\times/k^\times$, we define $\delta(\omega) = 1$ if ω is trivial, and $\delta(\omega) = 0$ otherwise.