Cambridge University Press
978-0-521-44804-8 - Shintani Zeta Functions
Akihiko Yukie
Excerpt
More information

Introduction

§0.1 **What is a prehomogeneous vector space?**

One contribution of Gauss to number theory in the early nineteenth century was the discovery of the correspondence between equivalence classes of integral binary quadratic forms and ideal classes of quadratic fields. This correspondence can be described as follows.

Let $f(v) = f(v_1, v_2) = x_0v_1^2 + x_1v_1v_2 + x_2v_2^2$ be a binary quadratic form such that x_0, x_1, x_2 are rational integers. We define an action of the group $\{\pm 1\} \times \mathrm{GL}(2, \mathbb{Z})$ on the set of integral binary quadratic forms so that if $g = (t, g_1)$ where $t = \pm 1, g_1 \in \mathrm{GL}(2, \mathbb{Z})$, $gf(v) = tf(vg_1)$. We consider equivalence classes of integral binary quadratic forms with respect to this action. It is easy to see that the discriminant $x_1^2 - 4x_0x_2$ is invariant under such an action. On the other hand, let m be a square free integer, and consider a non-zero ideal $\mathfrak{a}$ of the ring of algebraic integers in the field $k = \mathbb{Q}(\sqrt{m})$. The discriminant Δ_k of k is m if $m \equiv 1 \bmod 4$ and $4m$ if $m \equiv 2$ or $3 \bmod 4$. As a module over $\mathbb{Z}$, $\mathfrak{a}$ is generated by two elements, say α, β, because $\mathfrak{a}$ is a torsion free rank two module over $\mathbb{Z}$. Consider the binary quadratic form $f_{\mathfrak{a}}(v) = N(\mathfrak{a})^{-1}N(\alpha v_1 + \beta v_2)$, where $N(\mathfrak{a}), N(\alpha v_1 + \beta v_2)$ are the norms. It is easy to see that $f_{\mathfrak{a}}$ depends only on the ideal class of $\mathfrak{a}$. Moreover, it turns out that ideal classes of k correspond bijectively to equivalence classes of primitive integral binary quadratic forms with discriminant Δ_k by the map $\mathfrak{a} \to f_{\mathfrak{a}}$.

Gauss established this correspondence in [16], and the reader can see a modern proof in Theorem 4 [3, p. 142]. Here, we consider a natural question: why do we consider such a correspondence? One conceptual reason is that it gives us a parametrization of ideal classes of quadratic fields in terms of a group action on a vector space. We can use this parametrization to actually compute the class numbers of quadratic fields. But what we are interested in in this book is a more analytic question. In order to illustrate our purpose, let us describe the conjecture of Gauss.

Let h_d be the number of $\mathrm{SL}(2, \mathbb{Z})$-equivalence classes of primitive integral binary quadratic forms which are either positive definite or indefinite. Then Gauss conjectured the asymptotic property of the average of h_d. However, an integral form in the sense of Gauss is a form $x_0v_1^2 + 2x_1v_1v_2 + x_2v_2^2$ such that x_0, x_1, x_2 are integers. Here we consider $x_0v_1^2 + x_1v_1v_2 + x_2v_2^2$ such that x_0, x_1, x_2 are integers. With this understanding, we have the following asymptotic formula

(0.1.1)
$$\sum_{0<-d<x} h_d \sim \frac{\pi}{18\zeta(3)} x^{\frac{3}{2}},$$
$$\sum_{0<d<x} h_d \log \epsilon_d \sim \frac{\pi^2}{18\zeta(3)} x^{\frac{3}{2}}.$$

where $\epsilon_d = \frac{1}{2}(t + u\sqrt{d})$ and (t, u) is the smallest positive integral solution of the equation $t^2 - du^2 = 1$.

This conjecture was first proved by Lipschitz [42] for $d < 0$, and by Siegel [69] for $d > 0$, and much work has been done on the error term estimate also (see [65, pp. 44,45] for example). However, we are allowing all integers d here, and if $d = m^2d'$ and d' is a square free integer, $h_d, h_{d'}$ are related by a simple relation. Therefore, we are counting essentially the same object infinitely many times in (0.1.1). If k is a quadratic field over $\mathbb{Q}$, let h_k, R_k be the class number and the regulator respectively. If d is a square free integer, h_d is the number of ideal classes with respect to multiplication by elements with positive norms. Therefore, this h_d is slightly different from h_k of a number field of discriminant d even though they are closely related.

The problem of counting $h_k R_k$ of quadratic fields k was first settled by Goldfeld–Hoffstein [17] and was slightly generalized by Datskovsky [9] from our viewpoint recently. Here, we quote Datskovsky's statement for the simplest case.

$$\text{(0.1.2)} \qquad \sum_{\substack{0<-\Delta_k<x \\ [k:\mathbb{Q}]=2}} h_k \sim \frac{\zeta(2)}{3\pi} \prod_p (1-p^{-2}-p^{-3}+p^{-4}) x^{\frac{3}{2}},$$
$$\sum_{\substack{0<\Delta_k<x \\ [k:\mathbb{Q}]=2}} h_k R_k \sim \frac{\zeta(2)}{3} \prod_p (1-p^{-2}-p^{-3}+p^{-4}) x^{\frac{3}{2}},$$

where k runs through quadratic fields and Δ_k is the discriminant. Note that $\frac{\zeta(2)}{3} = \frac{\pi^2}{18}$. Therefore, (0.1.1) and (0.1.2) are very similar except for the difference between $\frac{1}{\zeta(3)}$ and $\prod_p (1-p^{-2}-p^{-3}+p^{-4})$.

Statements like (0.1.2) are the kind of density theorems we are looking for. We discuss the difference between (0.1.1) and (0.1.2) later in the introduction, and we go back to the space of binary quadratic forms again. The main ingredients of the above correspondence were the group $G = \mathrm{GL}(2)$ acting on the vector space V of binary quadratic forms, and the polynomial $\Delta(x) = x_1^2 - 4x_0x_2$ $(x = (x_0, x_1, x_2))$ which satisfies the property $\Delta(gx) = \det g \Delta(x)$ for $g \in \mathrm{GL}(2), x \in V$. Moreover, if we consider this vector space over an algebraically closed field, the generic point is a single G-orbit. The fundamental reason why one can prove results like (0.1.1), (0.1.2) is that we can use the Fourier analysis on the vector space V. Also when we consider the averages as (0.1.1) or (0.1.2), we can use the value of $\Delta(x)$ to average over. The fact that the generic point is a single orbit assures us that there is essentially one choice of such a polynomial.

Sato and Shintani introduced the notion of prehomogeneous vector spaces in [60] and generalized the situation as the above example. We now state the definition of prehomogeneous vector spaces from our viewpoint.

Let k be an arbitrary field. Let G be a connected reductive group, V a representation of G, and χ_V an indivisible non-trivial rational character of G, all defined over k.

Definition (0.1.3) *The triple (G, V, χ_V) is called a prehomogeneous vector space if the following two conditions are satisfied.*
(1) There exists a Zariski open orbit.
(2) There exists a polynomial $\Delta \in k[V]$ such that $\Delta(gx) = \chi'(g)\Delta(x)$ where χ' is a rational character and $\chi' = \chi_V^a$ for some positive integer a.

Note that if Δ_1, Δ_2 are two polynomials as in the above definition, there exist positive integers a, b such that $\frac{\Delta_1^a}{\Delta_2^b}$ is a G-invariant rational function. Since there exists an open orbit, this implies that $\frac{\Delta_1^a}{\Delta_2^b}$ is a constant function. Therefore, for any

k-algebra R, the set $V_R^{\rm ss} = \{x \in V_R \mid \Delta(x) \neq 0\}$ does not depend on the choice of Δ, and we call it the set of semi-stable points. A polynomial Δ which satisfies the property (2) is called a relative invariant polynomial. If V is an irreducible representation, the center of the image $G \to \mathrm{GL}(V)$ has split rank one by Schur's lemma. Therefore the choice of χ_V is unique. So we call (G, V) a prehomogeneous vector space also.

For the space of binary quadratic forms, let $G = \mathrm{GL}(1) \times \mathrm{GL}(2)$, and $\chi_V(t, g) = t \det g$ for $g = (t, g)$. Then (G, V, χ_V) is a prehomogeneous vector space in the above sense.

Before we start the discussion on prehomogeneous vector spaces, let us make one more historical remark.

There is no doubt that Gauss was the first mathematician who recognized the group theoretic approach to number theory. But one particular prehomogeneous vector space had appeared already in the eleventh century.

The solution of cubic and quartic equations by radicals has been known for a long time. But before the solution was found, there was a poet-mathematician Omar Khayyam in medieval Persia who worked on this problem. He did not think it was possible to solve cubic equations by radicals, and instead he tried to express the solutions to cubic equations geometrically. For example, the solution of the equation $x^3 = N$ can be realized as the intersection of two parabolas $y = x^2, y^2 = Nx$. After the solution by radicals was found, his work has long been forgotten. However, it is surprisingly related to the theory of prehomogeneous vector spaces.

What makes the space of binary quadratic forms so interesting is that we can associate a quadratic field to a generic point of the vector space. More precisely, if G is $\mathrm{GL}(1) \times \mathrm{GL}(2)$, V is the space of binary quadratic forms and $\chi_V(t, g) = t \det g$, there is a map from $G_{\mathbb{Q}} \setminus V_{\mathbb{Q}}^{\rm ss}$ to the set of isomorphism classes of fields of degree less than or equal to 2 over $\mathbb{Q}$. This map is clearly surjective, and this surjectivity is the reason why we count the class number of all the quadratic fields in (0.1.2).

Now, let us consider the group $G = \mathrm{GL}(3) \times \mathrm{GL}(2)$, and the vector space V of pairs of ternary quadratic forms. If we define $\chi_V(g_1, g_2) = (\det g_1)^4 (\det g_2)^3$, the triple (G, V, χ_V) is a prehomogeneous vector space (see [59] or Chapter 8). If $x = (Q_1, Q_2) \in V_k$ and Q_1, Q_2 are ternary quadratic forms, we can consider the set $\mathrm{Zero}(x) = \{v \in \mathbb{P}^2 \mid Q_1(v) = Q_2(v) = 0\}$. We call $\mathrm{Zero}(x)$ the zero set of x. We will show in Chapter 8 that (G, V, χ_V) is a prehomogeneous vector space and $V^{\rm ss}$ consists of points whose zero sets are four distinct points in $\mathbb{P}^2$. It is easy to see that the field generated by the residue fields of points in $\mathrm{Zero}(x)$ is a splitting field of a quartic equation. Now the question is if we can get all such fields from pairs of ternary quadratic forms. This is easy because any quartic equation $x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0$ can be written as an intersection of two conics as follows.

$$y = x^2, y^2 + a_1 xy + a_2 x^2 + a_3 x + a_4 = 0.$$

But this is what Omar Khayyam did about 900 years ago, and he essentially proved the surjectivity of the map from $G_{\mathbb{Q}} \setminus V_{\mathbb{Q}}^{\rm ss}$ to the isomorphism classes of splitting fields of quartic equations. For the works of Omar Khayyam, the reader should see [74]. The analytic theory of this prehomogeneous vector space is the main topic of this book, and we handle the global zeta function for this case in Part IV.

§0.2 The classification

In this section, we discuss the classification of irreducible prehomogeneous vector spaces over an algebraically closed field. Throughout this section, k is an algebraically closed field of characteristic zero.

First, we show that from a given prehomogeneous vector space, we can make infinitely many prehomogeneous vector spaces which are essentially the same as the original prehomogeneous vector space.

Let G be a reductive group, and V a representation of G. Suppose that the dimension of V is n. For an integer $0 < m < n$, we consider two representations $(G\times \mathrm{GL}(m), V\otimes k^m)$, $(G\times \mathrm{GL}(n-m), V\otimes k^{n-m})$. Then generic $\mathrm{GL}(m)_k$-orbits of $V_k\otimes k^m$ correspond bijectively with generic $\mathrm{GL}(n-m)_k$-orbits of $V_k \otimes k^{n-m}$ because the Grassmann variety of m planes in V can be identified with the Grassmann variety of $n-m$ planes in V. Therefore, $(G\times \mathrm{GL}(m), V\otimes k^m)$ is a prehomogeneous vector space if and only if $(G\times \mathrm{GL}(n-m), V\otimes k^{n-m})$ is a prehomogeneous vector space, and the sets of generic orbits coincide. If two prehomogeneous vector spaces are related in this way, we identify two such representations and consider the equivalence relation determined by this identification. If the dimension of (G, V) is the smallest among prehomogeneous vector spaces which are equivalent to (G, V), we say that (G, V) is reduced. Also we identify two prehomogeneous vector spaces $(G, V), (G', V)$ if the images of G, G' in $\mathrm{GL}(V)$ are the same.

Sato and Kimura proved in [59] that the following is the list of all the irreducible reduced prehomogeneous vector spaces.

(1) $G = \mathrm{GL}(n)\times H$, $V = M(n,n)_k$ where $H\subset \mathrm{GL}(n)$ is any reductive subgroup such that the k^n is an irreducible representation of H.

(2) $G = \mathrm{GL}(1)\times \mathrm{GL}(n)$, $V = \mathrm{Sym}^2 k^n$.

(3) $G = \mathrm{GL}(1)\times \mathrm{GL}(2n)$, $V = \wedge^2 k^{2n}$.

(4) $G = \mathrm{GL}(1)\times \mathrm{GL}(2)$, $V = \mathrm{Sym}^3 k^2$.

(5), (6), (7) $G = \mathrm{GL}(1)\times \mathrm{GL}(n)$, $V = \wedge^3 k^n$ where $n = 6, 7, 8$.

(8) $G = \mathrm{GL}(3)\times \mathrm{GL}(2)$, $V = \mathrm{Sym}^2 k^3\otimes k^2$.

(9) $G = \mathrm{GL}(6)\times \mathrm{GL}(2)$, $V = \wedge^2 k^6\otimes k^2$.

(10), (11) $G = \mathrm{GL}(n)\times \mathrm{GL}(5)$, $V = k^n\otimes \wedge^2 k^5$ where $n = 3, 4$.

(12) $G = \mathrm{GL}(3)\times \mathrm{GL}(3)\times \mathrm{GL}(2)$, $V = k^3\otimes k^3\otimes k^2$.

(13) $G = \mathrm{GSp}(2n)\times \mathrm{GL}(2m)$, $V = k^{2n}\otimes k^{2m}$ where $n\geq 2m\geq 2$.

(14) $G = \mathrm{GL}(1)\times \mathrm{GSp}(6)$, V is a 14 dimensional representation of G.

(15) $G = \mathrm{GO}(n)\times \mathrm{GL}(m)$, $V = k^n\otimes k^m$ where $n\geq 3$, $\frac{n}{2}\geq m\geq 1$.

(16), (17), (18) $G = \mathrm{GSpin}(7)\times \mathrm{GL}(n)$, $V = \mathrm{spin}_7\otimes k^n$ where $n = 1, 2, 3$ and spin_7 is the spin representation.

(19), (22) $G = \mathrm{GSpin}(n)$, $V = \mathrm{spin}_n$ where $n = 9, 11$.

(20), (21) $G = \mathrm{GSpin}(10)\times \mathrm{GL}(n)$, $V = \mathrm{halfspin}_{10}\otimes k^n$ where $\mathrm{halfspin}_{10}$ is the halfspin representation and $n = 2, 3$.

(23), (24) $G = \mathrm{GL}(1)\times \mathrm{GSpin}(n)$, $V = \mathrm{halfspin}_n$ where $n = 12, 14$.

(25), (26) $G = G_2\times \mathrm{GL}(n)$, $V = k^7\otimes k^n$ where k^7 is a representation of G_2 and $n = 1, 2$.

(27), (28) $G = E_6\times \mathrm{GL}(n)$, $V = k^{27}\otimes k^n$ where k^{27} is a representation of E_6 and $n = 1, 2$.

(29) $G = \mathrm{GL}(1)\times E_7$, V is a 56 dimensional representation of E_7

(30) $G = \mathrm{GSp}(2n)\times \mathrm{GO}(3)$, $V = k^{2n}\otimes k^3$.

The cases (1)–(29) are what we call regular prehomogeneous vector spaces. For the definition of the regularity, the reader should see [59]. Even though it does not make any difference over an algebraically closed field, we have included the GL(1) factor in (2)–(5) etc. and used groups like GSp$(2n)$, GO(n) instead of Sp$(2n)$, O(n) etc., because it is more natural number theoretically. Most of these representations are what we call prehomogeneous vector spaces of parabolic type classified by Rubenthaler in his thesis [52]. This is the kind of prehomogeneous vector spaces which one can construct from parabolic subgroups of reductive groups as follows.

Let G be a reductive group, and $P = MU$ a standard parabolic subgroup where M is the Levi component and U is the unipotent radical. The reductive part M acts on U by conjugation, and therefore on $V = U/[U,U]$ also. Since V can be considered as a vector space, we have a representation of a reductive group M. Vinberg [75] proved that there is a Zariski open orbit. Therefore, if there exists a relatively invariant polynomial, (M, V) is a prehomogeneous vector space by choosing a relative invariant polynomial and is called a prehomogeneous vector space of parabolic type.

For example, if we consider the Siegel parabolic subgroup P of GSp$(2n)$, $M = \mathrm{GL}(1) \times \mathrm{GL}(n)$ and V is the space of quadratic forms in n variables. If G is a type C_n group etc., we say that (M, V) is of type C_n etc. Then (2) is C_n type, (3) is D_{2n} type, (4) is G_2 type, (5), (6), (7) are of E_6, E_7, E_8 types, (8) is F_4 type, (9) is E_7 type, (10), (11) are E_7, E_8 types, (12) is E_6 type, (13) is C_{n+m} type, (14) is F_4 type, (15) is B, D type, (16) is F_4 type, (20), (23), (25), (27) are E_7 type, (21), (24), (26), (28) are E_8 type (29) is E_8 type. (1) is not always of parabolic type. (17), (18) (19), (22), (25), (26) are not in Table 1 [52, pp. 35–38].

For the details on prehomogeneous vector spaces of parabolic type, the reader should see [52].

§0.3 The global zeta function

In this section, we discuss the meromorphic continuation and the functional equation of the zeta function, restricting ourselves to irreducible prehomogeneous vector spaces (G, V, χ_V) for simplicity. The reader should see §3.1 for the general definition of the zeta function. For the rest of this section, k is a number field.

For simplicity, we assume that there exists a one dimensional split torus $T_0 \cong \mathrm{GL}(1)$ in the center of G acting on V by the ordinal multiplication by $t^{e_0} \in \mathrm{GL}(1)$ and $\chi_V(t) = t^e$ for $t \in T_0$ where $e_0, e > 0$ are positive integers. Let Δ be a relative invariant polynomial, and d the degree of Δ. Then $|\Delta(gx)| = |\chi_V(g)|^{\frac{e_0 d}{e}} |\Delta(x)|$. Let N be the dimension of V.

We assume that the representation $G \to \mathrm{GL}(V)$ is faithful. Therefore, in terms of the list in §0.2, we are considering $(G/\widetilde{T}, V)$ where $\widetilde{T}$ is the kernel of the homomorphism $G \to \mathrm{GL}(V)$. We fix a Haar measure dg on $G_{\mathbb{A}}$. Moreover, we assume that dg is of the form $dg = \prod_v dg_v$ where dg_v is a Haar measure on G_{k_v} for $v \in \mathfrak{M}$. Let $L \subset V_k^{\mathrm{ss}}$ be a G_k-invariant subset. For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$ and a complex variable s, we define

$$Z_L(\Phi, s) = \int_{G_{\mathbb{A}}/G_k} |\chi_V(g)|^{\frac{e_0 s}{e}} \sum_{x \in L} \Phi(gx) dg, \tag{0.3.1}$$

$$Z_{L+}(\Phi, s) = \int_{\substack{G_{\mathbb{A}}/G_k \\ |\chi_V(g)| \geq 1}} |\chi_V(g)|^{\frac{e_0 s}{e}} \sum_{x \in L} \Phi(gx) dg,$$

if these integrals are well defined.

We say that (G, V, χ_V) is of complete type if $Z_{V_k^{ss}}(\Phi, s)$ converges absolutely for $\mathrm{Re}(s) \gg 0$ and $Z_{V_k^{ss}+}(\Phi, s)$ is an entire function for all $\Phi \in \mathscr{S}(V_{\mathbb{A}})$. It is very likely that the first condition implies the second condition. We say that (G, V, χ_V) is of incomplete type if it is not of complete type. If the stabilizer G_x contains a split torus in its center for some $x \in V_k^{ss}$, (G, V, χ_V) is of incomplete type. This applies to the cases (2) $n = 2$, (12), (15) $m = 1, 2$, (17) in §2. If G_x does not contain a split torus in its center for any $x \in V_k^{ss}$, it is very likely that it is of complete type even though this has yet to be proved. Some examples are known. Siegel showed that the case (2) in §0.2 for $n \geq 3$ is of complete type. In general, if dim G = dim V, it is of complete type, as we show in this book. This applies to the cases (4), (8), (11) in §0.2 (the case (4) is due to Davenport and Shintani). If (G, V, χ_V) is an irreducible prehomogeneous vector space of complete type, we choose V_k^{ss} as L in the definition of the zeta function and use the notation $Z(\Phi, s)$, $Z_+(\Phi, s)$.

We fix a measure $dx = \prod_v dx_v$ on $V_{\mathbb{A}}$. Then

$$d(gx) = |\chi_V(g)|^{\frac{e_0 N}{e}} dx, \ |\Delta(gx)| = |\chi_V(g)|^{\frac{e_0 d}{e}} |\Delta(x)|.$$

First we show that $Z_L(\Phi, s)$ decomposes into a summation over rational orbits. Suppose $\Phi = \otimes_v \Phi_v \in \mathscr{S}(V_{\mathbb{A}})$. Let $x \in V_k^{ss}$. Let G_x be the stabilizer of x, and G_x^0 its connected component of 1. Let $o(x) = [G_{xk} : G_{xk}^0]$. Then by an obvious consideration,

$$Z_L(\Phi, s) = \sum_{x \in G_k \backslash L} \frac{1}{o(x)} \int_{G_{\mathbb{A}}/G_{xk}^0} |\chi_V(g)|^{\frac{e_0 s}{e}} \Phi(gx) dg.$$

Let $x \in L$, and $v \in \mathfrak{M}$. We choose a left G_{k_v}-invariant measure $dg'_{x,v}$ on $G_{k_v}/G_{xk_v}^0$ and a Haar measure $dg'_{x,v}$ on $G_{xk_v}^0$ so that $dg_v = dg'_{x,v} dg''_{x,v}$ for all v and $dg'_x = \prod_v dg'_{x,v}$, $dg''_x = \prod_v dg''_{x,v}$ are well defined. Then there exists a constant $b_{x,v} > 0$ such that

$$\int_{G_{k_v}/G_{xk_v}^0} \Psi(g'_{x,v} x) dg'_{x,v} = b_{x,v} \int_{G_{k_v} x} \Psi(y_v) |\Delta(y_v)|_v^{-\frac{N}{d}} dy_v$$

for any measurable function Ψ on $G_{k_v} x$. We discuss the choice of the measures $dg'_{x,v}, dg''_{x,v}$ for some cases in §0.5.

Let $\mu(x)$ be the volume of $G_{x\mathbb{A}}^0/G_{xk}^0$ with respect to the measure dg''_x. We define

$$X_{x,v}(\Phi, s) = \int_{G_{k_v} x} \Phi_v(y_v) |\Delta(y_v)|_v^{\frac{s-N}{d}} dy_v,$$

$$Z_{x,v}(\Phi, s) = \int_{G_{k_v}/G_{xk_v}^0} |\chi_V(g'_{x,v})|^{\frac{e_0 s}{e}} \Phi_v(g'_{x,v} x) dg'_{x,v}.$$

Note that these distributions depend only on the orbit $G_{k_v}x$ and Φ_v. These distributions are related as follows.

$$\begin{aligned} Z_{x,v}(\Phi,s) &= |\Delta(x)|^{-\frac{s}{d}} \int_{G_{k_v}/G^0_{xk_v}} |\Delta(x)|^{\frac{s}{d}} |\chi_V(g'_{x,v})|^{\frac{e_0 s}{e}} \Phi_v(g'_{x,v}x)dg'_{x,v} \\ &= |\Delta(x)|^{-\frac{s}{d}} \int_{G_{k_v}/G^0_{xk_v}} |\Delta(g'_{x,v}x)|^{\frac{s}{d}} \Phi_v(g'_{x,v}x)dg'_{x,v} \\ &= b_{x,v}|\Delta(x)|_{v_0}^{-\frac{s}{d}} X_{x,v}(\Phi,s). \end{aligned}$$

We choose an infinite place v_0. Then the above decomposition becomes

$$\begin{aligned} (0.3.2)\quad Z_L(\Phi,s) &= \sum_{x\in G_k\backslash L} \frac{\mu(x)}{o(x)} \prod_{v\in\mathfrak{M}} Z_{x,v}(\Phi,s) \\ &= \sum_{x\in G_k\backslash L} \frac{\mu(x)}{o(x)|\Delta(x)|_{v_0}^{\frac{s}{d}}} b_{x,v_0} X_{x,v_0}(\Phi,s) \prod_{v\in\mathfrak{M}\backslash\{v_0\}} Z_{x,v}(\Phi,s). \end{aligned}$$

Let $V_1,\cdots,V_l$ be $G_{k_{v_0}}$-orbits of $V^{\mathrm{ss}}_{k_{v_0}}$. Note that even though the group G is assumed to be connected as an algebraic group, $G_{k_{v_0}}$ may not be connected in general, for example GL(1). The distribution $X_{x,v_0}(\Phi,s)$ only depends on the sets $V_1,\cdots,V_l$, and we denote it by $X_1(\Phi,s),\cdots,X_l(\Phi,s)$. We define

$$\begin{aligned} (0.3.3)\qquad X(\Phi,s) &= (X_1(\Phi,s),\cdots,X_l(\Phi,s)), \\ \xi_{L,i}(\Phi,s) &= \sum_{x\in G_k\backslash V_i\cap L} \frac{\mu(x)b_{x,v_0}}{o(x)|\Delta(x)|_{v_0}^{\frac{s}{d}}} \prod_{v\in\mathfrak{M}\backslash\{v_0\}} Z_{x,v}(\Phi,s), \\ \xi_L(\Phi,s) &= \begin{pmatrix} \xi_{L,1}(\Phi,s) \\ \vdots \\ \xi_{L,l}(\Phi,s) \end{pmatrix}. \end{aligned}$$

Then

$$Z_L(\Phi,s) = X(\Phi,s)\xi_L(\Phi,s).$$

If $L = V^{\mathrm{ss}}_k$, we drop L and use the notation $\xi(\Phi,s)$ etc.

Let V^* be the dual space of V and (x,y) the natural pairing of $x\in V, y\in V^*$. For $g\in G$, let $g^*\in \mathrm{GL}(V^*)$ be the element such that $(gx,y)=(x,g^*y)$ for all $x\in V, y\in V^*$. Then $g\to (g^*)^{-1}$ is a representation of G on V^* and is called the contragredient representation of (G,V). If (G,V) is a prehomogeneous vector space, (G,V^*) is also a prehomogeneous vector space, and there exists a relative invariant polynomial Δ^* of the same degree d. Moreover $\chi_{V^*}(g)=\chi_V(g)^{-1}$. Also $V^*_{v_0}$ has the same number of $G_{k_{v_0}}$-orbits $V_1^*,\cdots,V_l^*$.

If $L^*\subset V^*_k$ is a G_k-invariant subset, we define functions $X_i^*(\Phi^*,s), \xi^*_{L^*,i}(\Phi^*,s)$ for $i=1,\cdots,l$ and $X^*(\Phi^*,s), \xi^*_{L^*}(\Phi^*,s)$ similarly as in (0.3.3) for V^*.

Let $\Delta^*(\partial),\Delta(\partial)$ be the differential operators with constant coefficients on V,V^* which correspond to Δ^*,Δ. The local theory at infinite places is based on the following fact.

Lemma (0.3.4) *There exists a polynomial $b(s)$ (resp. $b^*(s)$) such that*

$$\Delta^*(\partial)\Delta(x)^s = b(s)\Delta(x)^{s-1} \ (\textit{resp.}\ \Delta(\partial)\Delta^*(x)^s = b^*(s)\Delta^*(x)^{s-1})$$

for any natural number s.

This lemma follows from the fact that $\Delta(\partial), \Delta^*(\partial)$ are invariant differential operators. These polynomials $b(s), b^*(s)$ are called the b-functions for Δ, Δ^* respectively. For these facts, the reader should see [60], [64].

Once we know (0.3.4), the meromorphic continuation of $X_i(\Phi, s)$ follows by the relations

$$X_i(\Delta^*(\partial)\Phi, s) = \pm b(\tfrac{s-N}{d})X_i(\Phi, s-d) \qquad v_0 \in \mathfrak{M}_{\mathbb{R}},$$
$$X(\Delta^*(\partial)\overline{\Delta^*(\partial)}\Phi, s) = \pm b(\tfrac{s-N}{d})\overline{b(\tfrac{\bar{s}-N}{d})}X(\Phi, s-d) \quad v_0 \in \mathfrak{M}_{\mathbb{C}}.$$

For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$, we define its Fourier transform $\widehat{\Phi} \in \mathscr{S}(V^*_{\mathbb{A}})$ by

$$\widehat{\Phi}(y) = \int_{V_k} \Phi(x) < (x, y) > dx.$$

For $\Phi^* \in \mathscr{S}(V^*_{\mathbb{A}})$, we define its Fourier transform $\widehat{\Phi}^* \in \mathscr{S}(V_{\mathbb{A}})$ similarly. If Φ has a product form $\Phi = \otimes_v \Phi_v$, $\widehat{\Phi}$ has the product form $\widehat{\Phi} = \otimes_v \widehat{\Phi}_v$ where $\widehat{\Phi}_v$ is the Fourier transform of Φ_v with respect to $< (x, y) >_v$.

The following theorem was proved by Sato and Shintani in [60].

Theorem (0.3.5) (Sato–Shintani) *The distributions $X_i(\Phi, s), X_i^*(\Phi^*, s)$ can be continued meromorphically to the entire $s \in \mathbb{C}$ for all i, and satisfy a functional equation*

$$X^*(\widehat{\Phi}, N-s) = X(\Phi, s)C(s),$$

where $C(s) = (c_{ij}(s))_{1 \le i,j \le k}$ is a function which is meromorphic everywhere.

We do not depend on (0.3.5) in this book, so we explain only briefly why (0.3.5) should be true.

If $g \in G_{k_{v_0}}$ and $\Phi_1(x) = \Phi(gx)$,

$$X_i(\Phi_1, s) = |\chi_V(g)|^{-\frac{e_0 s}{e}} X_i(\Phi, s),\ X_i^*(\widehat{\Phi}_1, N-s) = |\chi_V(g)|^{-\frac{e_0 s}{e}} X_i^*(\widehat{\Phi}, N-s)$$

for all i.

Choose $x_i \in V_i$ for $i = 1, \cdots, l$. Then $V_i \cong G_{k_{v_0}}/G_{x_i, k_{v_0}}$. So if $\Phi \in \mathscr{S}(V_{k_{v_0}})$ is compactly supported, (0.3.5) follows from the uniqueness of left $G_{k_{v_0}}$-invariant measures on orbits $V_1, \cdots, V_l$. For the proof of (0.3.5) for general Φ_{v_0}, the reader should see [60].

A similar statement to the above theorem for reducible prehomogeneous vector spaces was proved by F. Sato. As we will see below, the functional equation of the global zeta function follows from (0.3.5) for complete types. Therefore, one can deduce a similar global functional equation under the assumption that the zeta function is defined for $V_k^{\rm ss}$. For this, the reader should see [56]–[58]. For finite places, a similar statement to the above theorem was proved by Igusa in [22].

Suppose that $(G, V), (G, V^*)$ are of complete type. We denote the zeta function for V^* by $Z^*(\Phi^*, s), Z_+^*(\Phi^*, s)$. Let $G^0_{\mathbb{A}} = \{g \in G_{\mathbb{A}} \mid |\chi_V(g)| = 1\}$. For $\Phi \in \mathscr{S}(V_{\mathbb{A}})$ and $\lambda \in \mathbb{R}_+$, we define $\Phi_\lambda \in \mathscr{S}(V_{\mathbb{A}})$ by the formula $\Phi_\lambda(x) = \Phi(\underline{\lambda} x)$. We choose a Haar measure dg^0 on $G^0_{\mathbb{A}}$ so that

$$Z(\Phi, s) = \int_0^\infty \int_{G^0_{\mathbb{A}}/G_k} \lambda^s \sum_{x \in V_k^{\rm ss}} \Phi(\underline{\lambda} g^0 x) d^\times \lambda dg^0.$$

For any $\Phi \in \mathscr{S}(V_{\mathbb{A}})$, we define

$$J(\Phi, g) = \left(|\chi(g)|^{-\frac{N}{e}} \sum_{x \in V_k^* \setminus V_k^{*ss}} \widehat{\Phi}((g^*)^{-1}x) - \sum_{x \in V_k \setminus V_k^{ss}} \Phi(gx) \right),$$
$$I^0(\Phi) = \int_{G^0_{\mathbb{A}}/G_k} J(\Phi, g^0) dg^0,$$
$$I(\Phi, s) = \int_0^1 \lambda^s I^0(\Phi_\lambda) d^\times \lambda.$$

Then by the Poisson summation formula,

(0.3.6) $$Z(\Phi, s) = Z_+(\Phi, s) + Z_+^*(\widehat{\Phi}, N - s) + I(\Phi, s).$$

Sato and Shintani proved the following theorem under a strong assumption in [60]. Shintani later used a slightly different argument and proved that the meromorphic continuation and the functional equation of the global zeta function follow from Theorem (0.3.5). The reader can see his argument in [65] where he only considers the space of quadratic forms, but his argument works for the general case as follows.

Theorem (0.3.7) *Suppose that $(G, V), (G, V^*)$ are of complete type. Then the zeta functions $Z(\Phi, s), Z^*(\Phi^*, s)$ can be continued meromorphically everywhere for all $\Phi \in \mathscr{S}(V_{\mathbb{A}}), \Phi^* \in \mathscr{S}(V^*_{\mathbb{A}})$ and satisfy a functional equation*

$$Z(\Phi, s) = Z^*(\widehat{\Phi}, N - s).$$

Proof. Consider $v_0 \in \mathfrak{M}_\infty$ in (0.3.2). Suppose $v_0 \in \mathfrak{M}_{\mathbb{R}}$. We fix $1 \le i \le l$ and prove that $\xi_i(\Phi, s)$ can be continued meromorphically everywhere. We choose $\Phi = \otimes_v \Phi_v$ so that $\Phi_{v_0}(x) \in C_0^\infty(V_i)$. Let $\Phi_1 = \otimes_v \Phi_{1v} \in \mathscr{S}(V_{\mathbb{A}})$ be the function such that $\Phi_{1v_0}(x) = \Delta^*(\partial)\Phi_{v_0}(x)$, and $\Phi_{1v} = \Phi_v$ for all $v \in \mathfrak{M} \setminus \{v_0\}$. Then $\widehat{\Phi}_{1v_0}(x) = (2\pi\sqrt{-1})^d \Delta^*(x)\Phi_{v_0}(x)$, so $\Phi_{1v_0}(x) = 0, \widehat{\Phi}_{1v_0}(y) = 0$ for $x \in V_{k_{v_0}} \setminus V^{ss}_{k_{v_0}}, y \in V^*_{k_{v_0}} \setminus V^{*ss}_{k_{v_0}}$. Then by (0.3.6), $Z(\Phi_1, s) = Z_+(\Phi_1, s) + Z_+^*(\widehat{\Phi}_1, N - s)$. Similarly, $Z^*(\widehat{\Phi}_1, s) = Z_+^*(\widehat{\Phi}_1, s) + Z_+(\Phi_1, N - s)$. Therefore, $Z(\Phi_1, s), Z^*(\widehat{\Phi}_1, s)$ are entire functions and $Z(\Phi_1, s) = Z^*(\widehat{\Phi}_1, N - s)$. Since $X_i(\Phi_1, s) = \pm b(\frac{s-N}{d}) X_i(\Phi, s - d)$, we can choose Φ_{v_0} so that $X_i(\Phi_1, s) \neq 0$. By the choice of Φ_{v_0}, $X_j(\Phi_1, s) = 0$ for $j \neq i$. Therefore, $X_i(\Phi_1, s)\xi_i(\Phi, s)$ is an entire function. This proves the meromorphic continuation of $\xi_i(\Phi, s)$. The meromorphic continuation of $\xi_i^*(\Phi^*, s)$ is similar.

By (0.3.5) and what we have just shown,

$$X^*(\widehat{\Phi}_1, N - s)\xi^*(\widehat{\Phi}, N - s) = X(\Phi_1, s)\xi(\Phi, s) = X(\Phi_1, s)C(s)\xi^*(\widehat{\Phi}, N - s).$$

Since $X_j(\Phi_1, s) = 0$ for $j \neq i$,

(0.3.8) $$\xi_i(\Phi, s) = \sum_{j=1}^{l} C_{ij}(s)\xi_j^*(\widehat{\Phi}, N - s).$$

Therefore,

$$\xi(\Phi, s) = C(s)\xi^*(\widehat{\Phi}, N - s).$$

Since $\xi(\Phi, s)$ and $\xi^*(\widehat{\Phi}, N - s)$ do not depend on the v_0 part, the above equation is true for any $\Phi \in \mathscr{S}(V_{\mathbb{A}})$.

Now we consider an arbitrary Φ. The meromorphic continuation of

$$Z(\Phi, s), Z^*(\Phi^*, s)$$

follows from that of $X(\Phi, s)$, $\xi(\Phi, s)$ etc. Using the relation (0.3.8),

$$\begin{aligned} Z(\Phi, s) &= X(\Phi, s)\xi(\Phi, s) = X(\Phi, s)C(s)\xi^*(\widehat{\Phi}, N - s) \\ &= X^*(\widehat{\Phi}, N - s)\xi^*(\widehat{\Phi}, N - s) \\ &= Z^*(\widehat{\Phi}, N - s). \end{aligned}$$

Thus the functional equation for general Φ follows.

If $v_0 \in \mathfrak{M}_{\mathbb{C}}$, the proof is similar, except that we consider either $\Phi_{1v_0}(x) = \Delta^*(\partial)\Phi_{v_0}(x)$ or $\overline{\Delta^*(\partial)}\Phi_{v_0}(x)$. Then

$$\widehat{\Phi_{1v_0}}(x) = (2\pi\sqrt{-1})^d\Delta^*(x)\Phi_{1v_0}(x) \text{ or } (2\pi\sqrt{-1})^d\overline{\Delta^*(x)}\Phi_{1v_0}(x).$$

It is easy to see that

$$X(\Phi_1, s) = \begin{cases} \pm b(\frac{s-N}{d}) \int_{V_{v_0}^{ss}} \Phi(y_{v_0})\Delta(y_{v_0})^{\frac{s-d-N}{d}}\overline{\Delta(y_{v_0})}^{\frac{s-N}{d}} dy_{v_0} \text{ or} \\ \pm\overline{b(\frac{\bar{s}-N}{d})} \int_{V_{v_0}^{ss}} \Phi(y_{v_0})\Delta(y_{v_0})^{\frac{s-N}{d}}\overline{\Delta(y_{v_0})}^{\frac{s-d-N}{d}} dy_{v_0}. \end{cases}$$

Therefore, $X(\Phi_1, s)$ is an entire function, and we can use the same argument.

Q.E.D.

Let Φ_1 be as in the proof of the above theorem. Let $v_0 \in \mathfrak{M}_{\mathbb{R}}$.

By the above consideration,

$$\xi_i(\Phi, s) = \frac{Z(\Phi_1, s)}{X_i(\Phi_1, s)} = \pm\frac{Z(\Phi_1, s)}{b(\frac{s-N}{d})X_i(\Phi, s-d)}$$

and $Z(\Phi_1, s)$ is an entire function. Therefore, if $s = s_0$ is a pole of $\xi_i(\Phi, s)$, it has to be a zero of $b(\frac{s-N}{d})X_i(\Phi, s - d)$. However, since Φ_{v_0} is compactly supported, $X_i(\Phi, s - d)$ is an entire function, and we can choose Φ_{v_0} so that $X_i(\Phi, s_0 - d) \neq 0$. Therefore, the order of the pole $s = s_0$ of $\xi_i(\Phi, s)$ does not exceed the multiplicity of $s - s_0$ in the polynomial $b(\frac{s-N}{d})$. If $v_0 \in \mathfrak{M}_{\mathbb{C}}$, the order of the pole $s = s_0$ does not exceed the order of multiplicity of $s - s_0$ in the polynomials $b(\frac{s-N}{d})$ and $\overline{b(\frac{\bar{s}-N}{d})}$ by a similar consideration.

Also by the above consideration, if $s = s_0$ is an order l pole of $\xi_i(\Phi, s)$, it is at most an order l pole of $Z(\Phi, s)$ by choosing some Φ_{v_0}.

Kimura and Ozeki computed b-functions for irreducible reduced regular prehomogeneous vector spaces (see [31] for example). Unfortunately, the orders of the poles and the multiplicities of the roots of $b(s)$ do not coincide, for example in the