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Chapter 1

Effective methods in $\mathcal{D}$-modules

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Motivations and introduction to the theory of $\mathcal{D}$-modules

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$\mathcal{D}$-modules : an overview towards effectivity

Motivations and introduction to the theory of $\mathcal{D}$-modules

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1. Introduction

Generally speaking, one describes a "$\mathcal{D}$-module" as a module over a ring of differential operators; for instance, we will work with the following ones:

– $A = \mathbf{C}[x, \partial]$, the ring of (linear) differential operators with polynomial coefficients; here $x = (x_1, \dots, x_n)$; $\partial = (\partial_1, \dots, \partial_n)$, $\partial_j = \frac{\partial}{\partial x_j}$. This ring can also be thought of as the ring of non-commutative polynomials in $2n$ variables x_i, ∂_i, with the "Heisenberg commutation relations" $[x_i, x_j] = 0$, $[\partial_i, \partial_j] = 0$, $[\partial_i, x_j] = \delta_{ij}$, the Kronecker symbol.

– $\mathcal{D} = \mathcal{O}[\partial]$, $\mathcal{O} = \mathbf{C}\{x\}$, the ring of convergent series; therefore $\mathcal{D}$ is the ring of differential operators with analytic coefficients at $0 \in \mathbf{C}^n$.

– $\widehat{\mathcal{D}} = \widehat{\mathcal{O}}[\partial]$, $\widehat{\mathcal{O}} = \mathbf{C}[[x]]$, the ring of formal power series.

More generally, it is often useful to work with sheaves of modules over a sheaf of rings of differential operators (e.g., over an open set of $\mathbf{C}^n$, or over an analytic manifold). For simplicity, we will not consider this case here.

Take e.g. the case of $\mathcal{D}$; the others are similar. A (left) $\mathcal{D}$-module can be thought of in two different ways.

i) A space of "functions" on which the differential operators act. Here, e.g. the space of $\mathcal{C}^\infty$ functions near 0 in $\mathbf{R}^n$; or the space of analytic functions at 0 in $\mathbf{R}^n$ or $\mathbf{C}^n$; or the space of germs at 0 of distributions in $\mathbf{R}^n$ *etc...* . This is normal, and the language of $\mathcal{D}$-modules will add nothing new here.

ii) A system of linear partial differential equations. This requires some explanation; denote by $\mathcal{C}$ any one of the spaces of functions considered above. A system of equations in $\mathcal{C}$ is given by a map $A\cdot : \mathcal{C}^p \to \mathcal{C}^q$, $A = (a_{ij})$, $1 \le i \le q$, $1 \le j \le p$, $a_{ij} \in \mathcal{D}$ (explicitly, such a system is $\sum a_{ij} f_j = g_i$, f_j and $g_i \in \mathcal{C}$).

To this system, we associate a left $\mathcal{D}$-module in the following way: denote by $\cdot A$ the map $\mathcal{D}^q \to \mathcal{D}^p$ defined by $b \mapsto bA$, b being the row vector $(b_1, \dots, b_q) \in \mathcal{D}^q$. Put $M = \operatorname{coker}(\cdot A)$, i.e. by definition, $\mathcal{D}^p/\mathcal{D}^q A$; as $\cdot A$ is clearly compatible with the structure of left $\mathcal{D}$-modules of $\mathcal{D}^p$ and $\mathcal{D}^q$, M is a left $\mathcal{D}$-module (but it has no natural structure of a right $\mathcal{D}$-module).

Denote by $\ker(A\cdot)$ the "homogeneous system" associated to A, i.e. the set of $f = \begin{pmatrix} f_1 \\ \vdots \\ f_p \end{pmatrix} \in \mathcal{C}^p$ such that $Af = 0$. The next proposition shows that $\ker(A\cdot)$ "depends only on M".

Proposition 1.1. — *One has an isomorphism* $\ker(A\cdot) \simeq \operatorname{Hom}_{\mathcal{D}}(M, \mathcal{C})$.

As usual, the right-hand side denotes the set of $\mathcal{D}$-linear maps $M \to \mathcal{C}$. To prove this one applies $\mathrm{Hom}_{\mathcal{D}}(\cdot, \mathcal{C})$ to the (exact) sequence $\mathcal{D}^q \to \mathcal{D}^p \to M \to 0$; this gives a sequence $0 \to \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{C}) \to \mathcal{C}^p \to \mathcal{C}^q$ (note that the map $u \mapsto u(1)$ gives an isomorphism $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, \mathcal{C}) = \mathcal{C}$). The last sequence is again exact by the so-called "left exactness" of $\mathrm{Hom}_{\mathcal{D}}(\cdot, \mathcal{C})$. See any treatise on homological algebra; or, better, verify by yourself that it is obvious!

Remark 1.2. — By using *resolutions* of M one can also express the equations with right-hand side in terms of M. I will not do it here, since it is a little bit more complicated and I will not need it. See in any treatise on homological algebra the general nonsense on "Hom" and "Ext^i".

Now, the question is: Why replace the simple notion of differential system by the complicated notion of $\mathcal{D}$-module? Of course, this is unnecessary to study a given equation as, e.g. the Laplace equation; the reasons are different:

i) One can have different "presentations" of M, e.g. different ways of represent it as a quotient as above. This gives different systems, whose solutions correspond to each other.

Actually these systems are, in a sense, "trivially equivalent" (as occurs, e.g. when one adds to the unknown functions some of their derivatives as new unknowns, adding also of course the corresponding equations). I leave it as a good exercise to the reader to make a precise statement. Anyway, the consideration of a $\mathcal{D}$-module gives a way of reasoning independent of the special system which represents it; this is sometimes useful.

ii) More generally, we can be interested, not in a special equation, given a priori, but in the general theory, and especially in some systems or classes of systems which would not be so easy to write explicitly; one tries to prove some general properties of such systems (these properties could eventually be useful for explicit calculations).

We will see several examples of such properties in the following sections. Let me give here only one example:

Take $f \in \mathcal{O}$, $f \neq 0$, and consider the ring $\mathcal{O}[f^{-1}]$ of meromorphic functions at $0 \in \mathbf{C}^n$ of the form g/f^k, $f \in \mathcal{O}$, $k \in \mathbf{N}$. This is obviously a $\mathcal{D}$-module. One can prove that it is generated over $\mathcal{D}$ by $1/f^k$, for $k \geq k_0$ (k_0 depending on f). For such a k, the map $a \mapsto a(\frac{1}{f^k})$ represents $\mathcal{O}[f^{-1}]$ as $\mathcal{D}/\mathcal{I}_k$, $\mathcal{I}_k$ the ideal annihilating $1/f^k$. Finding the corresponding presentation means just finding a basis of $\mathcal{I}_k$, which in general is not at all obvious!

[*Incidentally*: two left ideals $\mathcal{I} \subset \mathcal{D}$ and $\mathcal{J} \in \mathcal{D}$ can look very different and nevertheless give the same module. The simplest example is the following:

take $n = 1$, $M = \mathcal{O} = \mathbf{C}\{x\}$. Then $\mathcal{O}$ is generated on $\mathcal{D}$ by 1; therefore $\mathcal{O} = \mathcal{D}/\mathcal{I}$, $\mathcal{I} = \mathrm{Ann}(1)$; it is easy to verify that $\mathcal{I}$ is generated by ∂. On the other hand, one has also $\mathcal{O} = \mathcal{D}x = \mathcal{D}/\mathcal{J}$, $\mathcal{J} = \mathrm{Ann}(x)$; it is easy to verify that $\mathcal{J}$ is not generated by one element. A minimal system of generators is, e.g. $x\partial - 1$, ∂^2.]

iii) A third, and very important, reason to consider $\mathcal{D}$-modules is the following: in this language, it is possible to define some natural *operations* e.g. direct and inverse images.

But these operations, except in some simple cases, have no obvious definition directly in terms of differential systems; and the corresponding systems are difficult to calculate explicitly.

I shall say a few words on these operations in section 5.

2. Finiteness properties

I will keep the notations of the preceding section. I shall state the results for $\mathcal{D}$, but they are equally true for $\widehat{\mathcal{D}}$ and A.

Theorem 2.1. — *The ring $\mathcal{D}$ is left and right nœtherian.*

Recall that "left nœtherian" means the following: let M be a left $\mathcal{D}$-module "finite over $\mathcal{D}$", i.e. admitting a finite system of generators; then, any $\mathcal{D}$-submodule of M is also finite.

It is sufficient to prove the result for $M = \mathcal{D}$, i.e. to prove that left ideals are finite (hint: use induction on the number of generators of M). To prove this last result, we use the *filtration* of $\mathcal{D}$ by the degree of the operators: let us say that $a = \sum a_\alpha \partial^\alpha$ ($a_\alpha \in \mathcal{O}$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$) is of degree $\leq k$ if $a_\alpha = 0$ for $|\alpha| = \alpha_1 + \cdots \alpha_n > k$; denote by $\mathcal{D}_k$ the operators of degree $\leq k$. Then the graded ring $\mathrm{gr}\,\mathcal{D} = \oplus(\mathcal{D}_k/\mathcal{D}_{k-1})$ is commutative since $[\mathcal{D}_k, \mathcal{D}_k] \subset \mathcal{D}_{k+\ell-1}$; actually it is equal to $\mathcal{O}[\xi]$, $\xi = (\xi_1, \ldots, \xi_n)$, with $\xi_i = \mathrm{gr}\,\partial_i$. Now, the result is a consequence of the following statements:

i) The ring $\mathcal{O}$ is nœtherian (classical: preparation theorem, standard basis, *etc...* .).

ii) Since $\mathcal{O}$ is nœtherian, $\mathcal{O}[\xi]$ is nœtherian; this is a classical result of Hilbert.

iii) Since $\mathrm{gr}\,\mathcal{D}$ is nœtherian, $\mathcal{D}$ itself is left (and right) nœtherian.

This is also classical, but I will recall the proof. Let $\mathcal{I}$ be a left ideal of $\mathcal{D}$; the ideal $\mathrm{gr}\,\mathcal{I}$ of $\mathrm{gr}\,\mathcal{D}$ is constructed with the "principal symbols" of operators of $\mathcal{I}$ (explicitly, if $a = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$, the principal symbol, more precisely the principal symbol of order k, is $\sigma(a) = \sum_{|\alpha|=k} a_\alpha \xi^\alpha$). Now, by *ii)*, $\operatorname{gr}\mathcal{I}$ has a finite basis, $(\bar{a}_1, \dots, \bar{a}_p)$; we can suppose that a_i is homogeneous, say of degree m_i; let $a_1, \dots, a_p \in \mathcal{D}$, of degrees respectively $m_1, \dots, m_p$ with $\sigma(a_i) = \bar{a}_i$. We will be finished if we prove the following lemma.

Lemma 2.2. — *The a_i's generate $\mathcal{I}$ over $\mathcal{D}$.*

Let $b \in \mathcal{I}$, $\deg b \leq \ell$, and let $\sigma(b)$ be its principal symbol (of degree ℓ); we have $\sigma(b) = \sum \bar{c}_i \bar{a}_i$, $\bar{c}_i \in \operatorname{gr}\mathcal{D}$; by homogeneity, we can suppose that $\bar{c}_i$ is homogeneous, of degree $\ell - m_i$; choose $c_i \in \mathcal{D}$, $\deg c_i \leq \ell - m_i$, $\sigma(c_i) = \bar{c}_i$; then $b - \sum c_i a_i$ belongs to $\mathcal{I}$, and its degree is $\leq \ell - 1$. By induction, we get the result.

Remark 2.3. — About the effectiveness of these constructions, we note that the only point is the effectiveness in $\mathcal{O}$; if we work with A instead of $\mathcal{D}$, $\mathcal{O}$ is replaced by $\mathbf{C}[[x]]$ and one can use Gröbner bases.

Remark 2.4. — The converse of lemma 2.2 is false: if $(a_1, \dots, a_p)$ is a basis of $\mathcal{I}$, $(\sigma(a_1), \dots, \sigma(q))$ is not in general a basis of $\operatorname{gr}\mathcal{I}$ (even if we take the principal symbol of the "exact degree" of the a_i's). Here is a classical counterexample: take $\mathcal{I} = (\partial_1, \delta)$, with $\delta = x_1\partial_2 + x_2\partial_3 + \dots + x_{n-1}\partial_n$; then ξ_1 and $\operatorname{gr}\delta$ do not generate $\operatorname{gr}\mathcal{I}$.

Actually, one has $[\partial_1, \delta] = \partial_2$; then $\partial_2 \in \mathcal{I}$; similarly, $[\partial_2, \delta] = \partial_3$, then $\partial_3 \in \mathcal{I}$, and so on; finally, we find that $\mathcal{I}$ is the ideal generated by $\partial_1, \dots, \partial_n$, i.e. the ideal of all differential operators without constant term; then $\operatorname{gr}\mathcal{I} = (\xi_1, \dots, \xi_n)$.

In fact, Stafford proved that, for any n, all ideals of $\mathcal{D}$ have *two* generators ! But this is obviously false for $\operatorname{gr}\mathcal{D}$ (and it is even false if we restrict ourselves to $\operatorname{gr}\mathcal{I}$, where $\mathcal{I}$ an ideal of $\mathcal{D}$).

3. Dimension theory

Let M be a finite (left) $\mathcal{D}$-module; the preceding constructions extend to M in the following way:

We define a *filtration* of M as a sequence $M_0 \subset \dots \subset M_k \subset \dots$ of finite $\mathcal{O}$-submodules of M with the following properties:

i) $\cup M_k = M$

ii) $\mathcal{D}_k M_\ell \subset M_{k+\ell}$ for all $k \geq 0$, $\ell \geq 0$.

The filtration is said to be *good* if one has $\mathcal{D}_k M_\ell = M_{k+\ell}$ for $\ell \geq \ell_0$ and all k ($\ell \geq \ell_0$ and $k = 1$ is sufficient).

Examples.

i) The obvious filtration of $\mathcal{D}$ (as a left $\mathcal{D}$-module) is obviously good.

ii) Any finite $\mathcal{D}$-module admits a good filtration: let $(m_1, \dots, m_p)$ be generators of M; then the map $\mathcal{D}^p \to M$ defined by $(a_1, \dots, a_p) \mapsto a_1 m_1 + \cdots + a_p m_p$ is surjective. We just take the filtration of M as the quotient of the obvious filtration of $\mathcal{D}^p$; this is clearly a good filtration.

iii) If $\mathcal{I}$ is an ideal of $\mathcal{D}$, the filtration of $\mathcal{I}$ induced by the trivial filtration of $\mathcal{D}$ is good; it is an easy consequence of the arguments of the preceding section.

iv) More generally, one can prove, in a more or less similar way, the following result: if $N \subset M$ are finite $\mathcal{D}$-modules, any good filtration of M induces a good filtration of N ("property of the Artin-Rees type").

Now, take a finite $\mathcal{D}$-module M and a good filtration $\{M_k\}$ on it. Then $\operatorname{gr} M = \oplus(M_{k+1}/M_k)$ is a module over $\operatorname{gr}\mathcal{D} = \mathcal{O}[\xi]$, and this module is *finite* [exercise: verify that this is equivalent to the goodness of the filtration].

Let V be the *support* of $\operatorname{gr} M$, which can be defined as follows: let $\overline{\mathcal{I}}$ be the ideal of $\operatorname{gr}\mathcal{D}$ annihilating $\operatorname{gr} M$; then V is the "set" of zeroes of $\overline{\mathcal{I}}$ (more exactly, V is a germ of sets for the family of $U_x \times \mathbf{C}^n_\xi$, U an open neighborhood of $0 \in \mathbf{C}^n$: if we take generators $\bar{a}_1, \dots, \bar{a}_p$ of $\overline{\mathcal{I}}$, their common zeroes define a set in $U \times \mathbf{C}^n$ for some U; and the corresponding germ is obviously independent of the chosen generators; in what follows, I will slightly abuse the language and speak of V as a set). V is analytic in all the variables (x, ξ) and algebraic and homogeneous with respect to the variables ξ. We call it the "characteristic variety" of M and denote it by char M.

Theorem 3.1. — *The characteristic variety depends only on M, and not on the chosen filtration.*

i) First, we prove the result for two good filtrations $\{M_k\}$ and $\{M'_k\}$ which satisfy $M'_k \subset M_k \subset M'_{k+1}$. We have the exact sequences

$$0 \to M_k/M'_k \to M'_{k+1}/M'_k \to M'_{k+1}/M_k \to 0 \quad \text{and}$$
$$0 \to M'_{k+1}/M_k \to M_{k+1}/M_k \to M_{k+1}/M'_{k+1} \to 0.$$

Denote by A (resp. B) the $\operatorname{gr}\mathcal{D}$-modules $\oplus(M_k/M'_k)$ (resp. $\oplus(M'_{k+1}/M_k)$). The preceding exact sequences give exact sequences

$$0 \to A \to \operatorname{gr}' M \to B \to 0 \text{ and}$$
$$0 \to B \to \operatorname{gr} M \to A \to 0.$$

Now, well-known properties of supports give $\operatorname{supp}(\operatorname{gr}' M) = \operatorname{supp}(A) \cup \operatorname{supp}(B)$ and the same for $\operatorname{gr} M$; then the result follows.

ii) Now, let $\{M_k\}$ and $\{N_k\}$ be two good filtrations of M. There exists some ℓ such that one has $M_k \subset N_{k+\ell}$ and $N_k \subset M_{k+\ell}$ (this is an easy consequence of the goodness). The result is obvious for $\ell = 0$, since, in that case, the filtrations coincide; we will then prove the result by induction on ℓ.

For $\ell \geq 1$, put $M'_k = M_k \cap N_{k+\ell-1}$, $N'_k = N_k \cap M_{k+\ell-1}$; these new filtrations are obviously good. One has $M'_k \subset N'_{k+\ell-1}$, $N'_k \subset M'_{k+\ell-1}$; therefore, by induction, the filtrations $\{M'_k\}$ and $\{N'_k\}$ give the same characteristic variety. On the other hand, we have $M'_k \subset M_k \subset M'_{k+1}$ and the same with M replaced by N; therefore by *i)*, in the preceding assertion, we can replace $\{M'_k\}$ by $\{M_k\}$ and $\{N'_k\}$ by $\{N_k\}$; this completes the proof.

Examples.

i) If $M = \mathcal{D}/\mathcal{I}$, and we take the filtration quotient on M, then $\operatorname{gr} M = \operatorname{gr}\mathcal{D}/\operatorname{gr}\mathcal{I}$ (on $\mathcal{I}$, we take the induced filtration). V is the "set" of zeroes of $\operatorname{gr}\mathcal{I}$, or the "set" of zeroes of the principal symbols of elements of $\mathcal{I}$.

But, if $M = \mathcal{D}/\mathcal{I}'$, $\mathcal{I}'$ another ideal, we can have $\operatorname{gr}' M \neq \operatorname{gr} M$: look e.g. at the case studied in 1,ii, [...].

ii) Suppose we have an exact sequence $0 \to M' \to M \to M'' \to 0$; if we have a good filtration on M, we have seen that the induced filtration on M' is good; on the other hand, the filtration quotient on M'' is obviously good. With these choices, we have again an exact sequence $0 \to \operatorname{gr} M' \to \operatorname{gr} M \to \operatorname{gr} M'' \to 0$ (verification left as an exercise). Therefore, we have $\operatorname{char} M = \operatorname{char} M' \cup \operatorname{char} M''$.

In principle, the preceding properties could be used to determine the characteristic variety of a $\mathcal{D}$-module, e.g. by using induction on the number of generators. However, this will only work in very special cases; in general, to calculate a characteristic variety is a hard problem!

There are severe restrictions on varieties of zeroes of ideals in $\operatorname{gr}\mathcal{D}$ that are characteristic varieties of $\mathcal{D}$-modules. The general necessary and sufficient condition is not known; however, a very important condition is the *involutiveness.*

Recall that this means: if we have two functions f and $g \in \mathcal{O}[\xi]$, we define their *Poisson bracket* by the formula

$$\{f,g\} = \sum_1^n \left[\frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial \xi_i}\right] .$$

This is related to the theory of differential equations as follows: if a and b are elements of $\mathcal{D}$, of degree respectively k and ℓ, then $[a,b] = ab - ba$ is of degree

$\leq k+\ell-1$ and, taking the principal symbols of the corresponding degrees, one has $\sigma[a,b]=\{\sigma(a),\sigma(b)\}$ (verification left to the reader).

Now, let $\mathcal{I}$ be an ideal of $\mathcal{O}[\xi]$, and V its variety. Let $\widetilde{\mathcal{I}}$ be the ideal of *all* elements of $\mathcal{O}[\xi]$ vanishing on V; by the so-called "Nullstellensatz", $\widetilde{\mathcal{I}}$ is the *root* of $\mathcal{I}$, i.e. the set of all $a \in \mathcal{O}[\xi]$ such that $a^k \in \mathcal{I}$ for some k.

One says that V is involutive if $\widetilde{\mathcal{I}}$ is stable under Poisson bracket, in other words if one has $\{\widetilde{\mathcal{I}},\widetilde{\mathcal{I}}\} \subset \widetilde{\mathcal{I}}$. Then, one has the following result.

Theorem 3.2. — *If M is a $\mathcal{D}$-module, then* $\operatorname{char} M$ *is involutive.*

This theorem is quite natural, since, if $\mathcal{I}$ is an ideal of $\mathcal{D}$, one has obviously $[\mathcal{I},\mathcal{I}] \subset \mathcal{I}$; therefore, taking the principal symbols, one has $\{\operatorname{gr}\mathcal{I},\operatorname{gr}\mathcal{I}\} \subset \operatorname{gr}\mathcal{I}$; but this does not prove the result at all: what we want is $\{\widetilde{\operatorname{gr}\mathcal{I}},\widetilde{\operatorname{gr}\mathcal{I}}\} \subset \widetilde{\operatorname{gr}\mathcal{I}}$, $\widetilde{\operatorname{gr}\mathcal{I}}$ the root of $\operatorname{gr}\mathcal{I}$, and this is more difficult.

All the proofs use microlocalization, in one way or another; I refer to the literature.

An important consequence of this theorem is the following: if an analytic set X is involutive, at each of its points its dimension is $\geq n$ (see appendix at the end of this section; I recall that the smooth part X^{reg} of X, i.e. the set of points where X is a non-singular complex manifold, is dense in X; and the dimension of X is by definition the complex dimension of X^{reg}). If we call the dimension of $\operatorname{char} M$ the *dimension* of M, then we have

Corollary 3.3. — *If M is a $\mathcal{D}$-module, its dimension is $\geq n$.*

A very interesting case is the case where $\dim M = n$; in that case, M is said to be *holonomic* and its characteristic variety *lagrangian* (see appendix for this last notion).

Examples of holonomic modules.

i) $\mathcal{O}$ as $\mathcal{D}$-module: in fact, we have a surjection $\mathcal{D} \to \mathcal{O}$ by $a \mapsto a(1)$. The kernel is the ideal generated by $(\partial_1,\ldots,\partial_n)$; therefore, $\operatorname{char}\mathcal{O}$ is the set $\xi_1=\cdots=\xi_n=0$.

ii) $\mathcal{D}/\mathcal{I}$, $\mathcal{I}$ an ideal generated by $x_1,\ldots,x_n$; the canonical generator (= image of $1 \in \mathcal{D}$) is usually called the "Dirac function" and denoted by δ, because the Dirac "function", or better distribution, is precisely annihilated by the x_i's. Of course, here, we have an "abstract" Dirac function, which is not in any sense a function or a distribution.

iii) For $f \in \mathcal{O}$, $f \neq 0$, one can prove that $\mathcal{O}[f^{-1}]$ is holonomic (see section 1, *ii)*). More generally, if M is holonomic, $M[f^{-1}]$ is holonomic. This is a difficult result due to Kashiwara. But the determination of the characteristic variety of $M[f^{-1}]$ is a very hard problem, only solved at the moment (by Ginzburg and Sabbah independently) in the case where M has "regular singularities". Note also that, if M is only finite over $\mathcal{D}$, then $M[f^{-1}]$ is not finite in general; counterexample, $M = \mathcal{D}$.

Appendix to section 3 : Symplectic geometry and involutive varieties

Let U be an open set in $\mathbf{C}^n$, with coordinates $(x_1, \dots, x_n)$; an analytic subset W of $U \times \mathbf{C}^n$ is called *involutive* if the following condition is satisfied: let $a \in V$, and let f and g be two holomorphic functions near a, vanishing on V; then $\{f, g\}$, defined as before, vanishes also on W.

The definition given here seems a little bit more restrictive than the one used before, since here we work with *local* equations and not only with global ones; but, if M is a finite $\mathcal{D}$-module, then $\operatorname{char} M$ is also involutive in the stronger sense: this would follow from the general theory of holomorphic functions; but, actually the proof of 3.2 gives the stronger result directly.

Let $a \in U \times \mathbf{C}^n$; for $\alpha, \alpha' \in T_a^*(U \times \mathbf{C}^n)$, the space of differentials at a, we define $\{\alpha, \alpha'\}_a$ as follows.

If $a = \sum_1^n \alpha_i dx_i + \sum \beta_i d\xi_i$, $\alpha' = \sum(\alpha_i' dx_i + \beta_i' d\xi_i)$, we put $\{\alpha, \alpha'\}_a = \sum(\beta_i \alpha_i' - \alpha_i \beta_i')$; therefore, for f and g holomorphic near a, we have $\{f, g\}(a) = \{df, dg\}_a$.

Now, take an involutive analytic set $W \subset U \times \mathbf{C}^n$; if a is a point of W^{reg}, the regular part of W, the *conormal* N_a at a to W (i.e. the subset of $T_a^*(U \times \mathbf{C}^n)$ for α satisfying $\alpha | W = 0$) is generated by differential of functions vanishing on W, because of the definition of W^{reg}; therefore, for $\alpha, \alpha' \in N_\alpha$, one has $\{\alpha, \alpha'\}_a = 0$. Using the notation $N_a^\perp$ orthogonal of N_a for the bilinear alternating form $\{\ ,\ \}_a$, this means that one has $N_a^\perp \subset N_a$. Now, as this form is obviously nondegenerate, one has $\dim N_a^\perp = 2n - \dim N_a$. Therefore one has $\dim N_a \leq n$, on $\dim_a W \geq n$. This proves the statement on the dimension used at corollary 3.3.

There is another way to look at this question, which is a little bit longer, but more illuminating. Put $\lambda = \sum \xi_i dx_i$, $\omega = d\lambda = \sum d\xi_i \wedge dx_i$.

For $a \in U \times \mathbf{C}^n$, ω gives an alternating two-form Ω on $T_a(U \times \mathbf{C}^n)$, by $\Omega(X, Y) = \langle X \wedge Y, \omega \rangle$; this form is non-degenerate, and therefore gives an isomorphism $\widetilde{\Omega} : T_a(U \times \mathbf{C}^n) \to T_a^*(U \times \mathbf{C}^n)$ by $\langle \widetilde{\Omega}(X), Y \rangle = \Omega(X, Y)$ or equivalently $\widetilde{\Omega}(X) = X \lrcorner\ \omega$ ("$\lrcorner$", is the interior product).

If f is a holomorphic function on $U \times \mathbf{C}^n$, the vector field $H_f = \widetilde{\Omega}^{-1}(df)$ is called the Hamiltonian vector field of f: this is the field corresponding