Chapter I

Geometric Aspects of Two-Dimensional Complexes

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The aim of this introductory chapter is to provide a geometric background for the algebra and homotopy theory to follow. We also focus on geometric questions of intrinsic interest. The algebraic topology of later chapters is meant to contribute to their understanding and treatment. What we present is an extract of courses the authors have given on Low Dimensional Topology. These courses were enriched by selected topics from further chapters of this book and/or some of the material which this article only summarizes, together with references to other sources.

1 Complexes of Low Dimensions and Group Presentations

A crucial tool for dealing with geometric and homotopy theoretic problems in topology is the decomposition of certain spaces into disjoint unions of *cells* e_i^n of various dimensions, each (open) cell e_i^n as a subspace being homeomorphic to an open unit disc $\mathring{D}^n = \{x \in \mathbb{R}^n \mid |x| < 1\}$. Thus, we may obtain the structure of a *cell complex* K for the underlying topological space $|K|$. We denote by K^n (the *n-skeleton*) the subspace comprised of the cells of K of dimensions $\leq n$ with the induced cell structure. It is also standard terminology to indicate by the superscript n of K^n that K is a complex with cells of at most dimension n (sometimes it is required that there *are* n-cells in an n-dimensional complex K).

The various notions of complexes differ by the conditions that are imposed on the closure $\overline{e}_i \subseteq |K|$ of a cell $e_i \in K$ and/or the *boundary* $\partial e_i = \overline{e}_i - e_i$. These conditions also regulate how the cells are "glued" together to yield the topology of $|K|$.

We assume that the reader is familiar with *simplical complexes* (finite or infinite). Here each closed cell $\overline{e}_i$ is a union of cells and is homeomorphic to a simplex with all its faces by a homeomorphism which maps open cells to open cells. (A homeomorphism between cell-complexes which maps open cells to open cells is called *cellular* or an *isomorphism*.) As a "gluing condition" one mostly confines to the *weak topology* with respect to the $\overline{e}_i^n$, i.e.,

(1) a subset of $|K|$ is closed iff its intersection with each $\overline{e}_i^n$ is closed.

Between 1939 and 1950, J.H.C. Whitehead published several papers which, on one hand, contain masterpieces of simplical techniques. Some of these are basic for our geometric questions and will be cited in several sections of this article. On the other hand, Whitehead gradually was led to the insight that many of his results in homotopy theory actually hold for a generalization of simplicial complexes, where proofs and constructions can avoid a lot of hard work (checking the strong conditions of simplicial complexes and maps). In [Wh49$_1$], he introduced the notion of "CW-complexes," for which the assumptions on the ∂e_i are far less restrictive than in the simplicial case. We give an inductive definition of CW-complexes which is particularly convenient in low dimensions. For the equivalence to Whitehead's original definition, see Schubert ([Schu64], III. 3, Exercise 1 and Sieradski ([Si92], Chapter 15).

1.1 Inductive construction of CW-complexes

Definition: A *CW-complex* K is a space $|K|$ with a cell decomposition, whose skeleta are inductively constructed as follows:

(a) K^0 is a discrete space, each point being a 0-cell.

(b) K^n is obtained by attaching to K^{n-1} a disjoint family D_i^n of closed n-discs via continuous functions $\varphi_i : \partial D_i^{n-1} \to K^{n-1}$, i.e.: take the topological sum $K^{n-1} + \sum D_i^n$ and pass to the quotient space given by the identifications $x \sim \varphi_i(x)$, $x \in \partial D_i^n$. Each $\overset{\circ}{D}{}_i^n$ then projects homeomorphically to an n-cell e_i^n. φ_i is called an *attaching map* for e_i^n.

(c) $|K| = \bigcup_{n=0}^{\infty} |K^n|$ is assigned the weak topology with respect to the $\overline{e}_i^n$ (as in (1)).

More generally, a cell complex is called a CW-complex, if it is isomorphic to one obtained by the preceding construction[1].

Remarks: (c) can be verified inductively for the skeleta (Exercise), hence it holds automatically if K is of finite dimension. The "W" in CW is motivated by Weak topology, the "C" (Closure finite) by the fact that each $\overline{e}_i^n$ is contained in a finite union of cells (Exercise).

In contrast to the situation of simplicial complexes, $\overline{e}_i^n$ itself is not necessarily a union of cells; see Figure 1, where ∂e^2 is a point of e^1, but not a 0-cell:

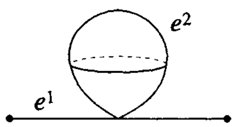

Figure I.1.

The following facts on CW-complexes can also be found in [Schu64] and [Si92]:

(2) A CW-complex K is finite (i.e., consists of finitely many cells) iff $|K|$ is compact.

(3) A covering space of a CW-complex K can be (uniquely) decomposed as a CW-complex $\tilde{K}$ such that the projection map $\tilde{K} \to K$ sends each cell $\tilde{e} \in \tilde{K}$ homeomorphically to a cell $e \in K$. (Note that the corresponding statement for closed cells is false in general, as the universal covering space of $S^1 = e^0 \cup e^1$ shows.) An attaching map for $\tilde{e}^n$ is obtained by appropriately lifting an attaching map $\varphi : \partial D^n \to K^{n-1}$ of e^n to $\tilde{K}^{n-1} = p^{-1}(K^{n-1})$.

We will deal mainly with finite CW-complexes and infinite ones that arise as covering complexes of (finite) complexes.

[1]Attaching maps in this general case are those of an isomorphic model according to (a), (b), (c), composed with a cellular homeomorphism. We shall tacitly assume similar extensions of definitions to be made later on. Note that specific attaching maps are not considered data of the complex; compare § 2.1.

1.2 Questions of subdivision and triangulation

A 1-dimensional CW-complex is a graph. It may contain loops and more than one edge between two vertices; but by introducing new vertices, it can be subdivided[2] to become simplicial (details left as an exercise):

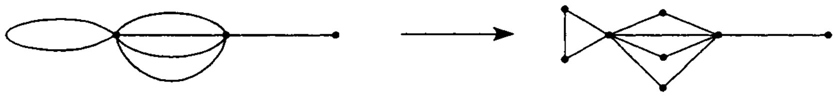

Figure I.2.

However, already in dimension 2 – due to the generality of attaching maps – there exist nontriangulable CW-complexes: Consider the finite(!) 2-complex K^2 of Figure 3, an *infinitely crumpled curtain* with three 0-cells, three 1-cells and one 2-cell, the attaching map of which oscillates on e_1^1:

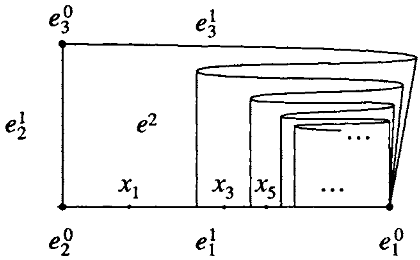

Figure I.3.

Note that all other marked points (e.g., the x_i) and lines of the drawing don't indicate further cells.

Theorem 1.1 *The CW-complex K^2 of Figure 3 is not homeomorphic to any simplicial complex.*

Proof: Triangulating appropriate neighbourhoods of the points x_i, $i = 1, 3, 5, 7, \ldots$, one obtains the local homology groups $H_2(|K|, |K| - \{x_i\}) = \underbrace{\mathbb{Z} \times \ldots \times \mathbb{Z}}_{(i-1)\text{ factors}}$. The number of "sheets" of the curtain coming together at x_i

[2] The general notion of subdivision for CW-complexes will be introduced below; see also footnote 5.

is thus seen to be an invariant of the local homeomorphism type of $|K|$ at x_i : $|K|$ has *infinitely many* distinct local homeomorphism types. Any triangulation of $|K|$ would have to be finite, as $|K|$ is compact (see (2)). But a finite simplicial complex has only *finitely many* local homeomorphism types, given by the stars of the simplices. Thus no such triangulation exists. □

In the beginning of this century, the "Hauptvermutung" of combinatorial topology was raised, the question, whether homeomorphic simplicial complexes are combinatorially equivalent (i.e., if they become isomorphic after simplicial subdivisions). The terminology naturally generalizes to CW-complexes: A *subdivision* K' of a CW-complex K is a CW-complex K' with $|K'| = |K|$ and the property that each cell $e \in K$ is the union of certain cells $e'_i \in K'$. K and L are *combinatorially equivalent*, if they admit subdivisions K' (of K) and L' (of L) which are isomorphic. But the answers to the Hauptvermutung question are different in the simplicial and the CW-case:

After contributions of prominent mathematicians (e.g., Papakyriakopoulos, Moise, Bing, Milnor, Stallings, Kirby, Siebenmann), the answer to the simplicial Hauptvermutung is known to be "Yes" for (locally finite[3] simplicial complexes of) dimensions ≤ 3; see [Br69], and "No" in dimensions ≥ 4. (Closed 4-manifolds with exotic differentiable structures were first exhibited by M. Kreck [Kr84$_2$]; that such examples yield combinatorially distinct triangulations follows from [Mu60] together with [Ce68]).

But the Hauptvermutung for CW-complexes fails already in dimension 2, as the following example(s) will show:

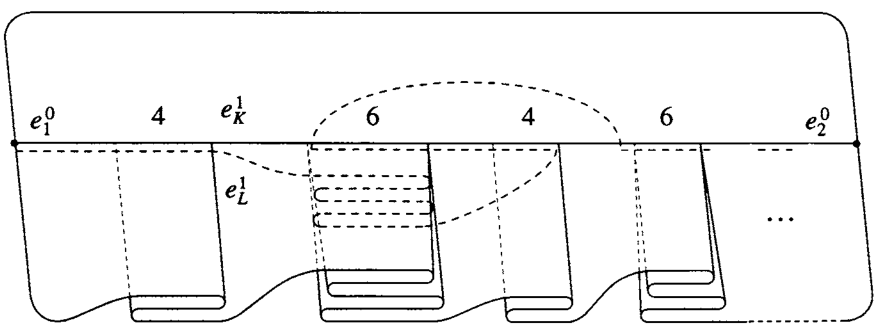

Figure I.4.

[3] A CW-complex is *locally finite*, if each point has a neighbourhood which meets only finitely many cells. (For simplicial complexes, this is equivalent to saying that the star of every simplex is finite (Exercise).) K is locally finite iff $|K|$ is locally compact; see [Schu64].

In Figure 4, K and L are finite CW-complexes with two 0-cells, three 1-cells and two 2-cells. They differ in their dissection of $|K| = |L|$ by the middle 1-cell e^1_K resp. e^1_L. The cell e^1_K contains a sequence of (open) subintervals, where four or six sheets of 2-dimensional material meet, the sequence converging towards e^0_2 alternatingly: $4, 6, 4, 6, \ldots$; e^1_L is defined by the broken line and its periodic and shrinking continuation towards e^0_2 with the "rhythm" $4, 4, 6, 6, 4, 4, 6, 6, \ldots$ of subintervals. It requires a little visualization to make sure that for the resulting 2-cells of L – above and below e^1_L – none of the CW-conditions are violated.

Theorem 1.2 *K^2 and L^2 of Figure 4 are homeomorphic but combinatorially inequivalent.*

Proof: We use some easy facts on the sheets, which result from local homology considerations as in the proof of Theorem 1.2.

a) Any subdivision K' of K, which is finite by (2), contains a 1-cell $e^1_{K'}$ (the one adjacent to e^0_2) characterized by the fact that almost all of the 4- and 6- sheeted subintervals are carried by it; L' analogously contains an $e^1_{L'}$.

b) Any homeomorphism of $|K|$ to $|L|(= |K|)$ must map a 4- resp. 6-sheeted subinterval to a 4- resp. 6-sheeted subinterval. Remembering a), a cellular homeomorphism of K' to L' thus in particular would have to map $e^1_{K'}$ to $e^1_{L'}$.

c) In e^1_K – hence also in $e^1_{K'}$ – any two subintervals of type 4 are separated by a type 6 subinterval (and vice versa), whereas the different "rhythm" $4, 4, 6, 6, \ldots$ in e^1_L implies that $e^1_{L'}$ contains adjacent subintervals of one type without such a separation by one of the other type. The different rhythms thus contradict the monotony of a potential homeomorphism $e^1_{K'}$ to $e^1_{L'}$. Hence there is no cellular homeomorphism $K' \to L'$. □

We refer to [Me67] for further subdivision phenomena, for instance:

(4) The relation of combinatorial equivalence for CW-complexes is not transitive.

Geometric pathologies arising from general attaching maps don't deserve too much interest of their own. But their occurrence either suggests the restriction to piecewise linear CW-complexes (see 1.4 and § 3 below) or it forces specific care in the proof of certain statements (see § 2.3 below).

1.3 Reading off presentations for π_1 of a CW-complex

The following material may be found in many textbooks. Proofs are based either on cellular/simplicial approximation (e.g., [Schu64]) or on the Seifert-van Kampen theorem (e.g. [Ma67], [Si92]); see also Chapter II, § 1 and § 2.

(5) If K^1 is connected[4], select a vertex e^0 as a basepoint and a *spanning tree*, i.e., a tree that consists of some edges and all vertices. Each remaining edge e_i^1, together with an orientation, determines a closed path from e^0 as in Figure 5 (by connecting the initial and terminal vertex of e_i^1 with e^0 on the tree). The elements a_i of $\pi_1(|K^1|, e^0)$ given by these paths constitute a free basis of $\pi_1(|K^1|, e^0)$, i.e., $\pi_1(|K^1|, e^0)$ is the free group $F(a_i)$.

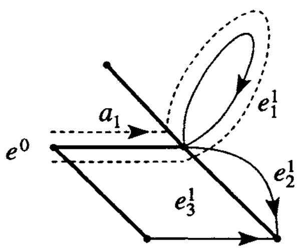

Figure I.5. Generating loops

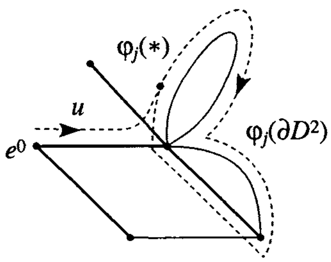

Figure I.6. Defining relations

(6) If K^2 is connected, select a basepoint, a spanning tree and paths a_i for the 1-skeleton as in (5). For each 2-cell e_j^2, an attaching map $\varphi_j : \partial D^2 \to K^1$, together with a base point $*$, an orientation of ∂D^2 and a connecting arc u from e^0 to $\varphi_j(*)$ (as in Figure 6) defines an element $R_j \in \pi_1(|K^1|, e_0)$ – a word R_j in the $a_i^{\pm 1}$ – which is trivial in $\pi_1(|K^2|, e^0)$. Moreover, $\pi_1(|K^1|, e^0) \to \pi_1(|K^2|, e^0)$ is surjective with the normal closure $N(R_j)$ (j ranging over all 2-cells) as the kernel.

Thus $\pi_1(|K^2|, e^0)$ has the *presentation* $\langle a_1, a_2, \ldots | R_1, R_2, \ldots \rangle$, i.e., the fundamental group $\pi_1(|K^2|, e^0)$ is given by *generators* a_i and *defining relations* R_j as the quotient $F(a_i)/N(R_j)$.

(7) If K has K^2 as its 2-skeleton, then the natural map $\pi_1(|K^2|, e^0) \to \pi_1(|K|, e^0)$ is an isomorphism.

[4] A CW-complex is connected iff its 1-skeleton is pathwise connected (Exercise).

A crucial point of this chapter (and book) is that a finite, connected 2-complex not only determines a fundamental group, but – via (6) – a certain class of presentations, which doesn't contain all presentations of π_1. Many homotopy invariants can be derived from any member of this presentation class. The first one to be mentioned is the Euler characteristic $\chi(|K|) = 1 - \#(a_i) + \#(R_j)$; see § 2.3.

As an example, note that for a compact, connected and simply connected K^2 the following statements are equivalent: (a) $|K|$ is *contractible* (i.e., $|K| \simeq *$); (b) $\chi(|K|) = 1$; (c) $H_2(|K|) = \{0\}$; (d) the presentations read off from $|K|$ according to (6) are *balanced*, i.e., the number of defining relations equals the number of generators (Exercise).

1.4 *PLCW*-complexes

In his papers [Wh39] and [Wh41$_2$], Whitehead didn't use general CW–complexes, but he confined his attention to specific ones which still admit simplicial[5] subdivisions, so–called "membrane complexes". They combine the advantage of a) providing complexes with "few cells" for a polyhedron with b) the piecewise linear (= p.l.) geometry, a main tool when embedding polyhedra into manifolds; see § 3. We refer to [Hu69], [RoSa72] and [Ze63-66] for the p.l. category, which is the proper home for simplicial theory when specific triangulations become irrelevant. In this category, membrane complexes are just what is given by the following definition:

Definition: A *PLCW-complex* is a locally finite CW-complex K together with a p.l. structure for $|K|$ such that

(a) all closed cells and all skeleta are subpolyhedra;

(b) K^n is obtained from K^{n-1} by a family of p.l. attaching maps $\varphi_i : \partial D_i^n \to K^{n-1}$ such that each subpolyhedron $e_i \cup K^{n-1}$ is p.l. homeomorphic, rel. K^{n-1}, to $D_i^n \underset{\partial D_i^n}{\cup} C(\varphi_i)$, where $C(\varphi_i)$ is the p.l. mapping cylinder of φ_i.

Here D^n is meant to be equipped with the p.l. structure given by a fixed homeomorphism of D^n to the n-cube I^n. b) involves results on p.l. mapping cylinders (existence, triangulations) which date back to [Wh39]; see the

[5]A *simplicial subdivision* K' of a CW-complex K is a subdivision of K by a simplicial complex K'.

discussion and the references in ([CoMeSa85], § 2). They yield, in particular, that by attaching finitely many n-cells to a $PL\,CW$-complex K^{n-1} via p.l. attaching maps, the result naturally becomes a $PL\,CW$-complex. Hence there is an inductive construction yielding finite $PL\,CW$-complexes. Exercise: Derive concrete triangulations in the 2-dimensional cases (8), (9) which follow.

Some classes of finite 2-dimensional PL CW-complexes:

(8) *Reidemeister complexes* (see [Re32]): For the construction of K^2 from K^1, let each edge of K^1 be equipped with a linear structure; each ∂D_j^2 is subdivided as a polygon; $\varphi_j : \partial D_j^2 \to K^1$ maps each edge linearly onto an edge of K^1 or onto a vertex. φ_j thus defines a closed *edge path* in K^1. The notion of a Reidemeister complex agrees with the one of a combinatorial CW-complex of dimension 2; see Chapter II, § 1.2.

After subdividing K^1, any (finite) $PL\,CW$-complex becomes a Reidemeister complex (Exercise). Simplicial 2-complexes are Reidemeister complexes in which each triangle of a simplicial K^1 may be filled in at most once.

(9) *Standard complexes of (finite) presentations:* As an "inverse" process to 1.3, we may associate to a finite presentation $\mathcal{P} = \langle a_1, \ldots, a_g | R_1, \ldots, R_h \rangle$ a Reidemeister-complex $K_{\mathcal{P}}$ with one vertex. Its 1-skeleton is a bouquet of circles with an oriented e_i^1 for each a_i:

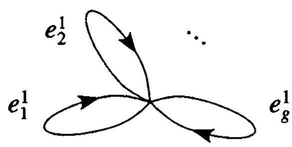

Figure I.7.

The 2-cells e_j^2 correspond bijectively to the R_j, which determine closed edge paths φ_j as the corresponding attaching maps.

Note that the sequence $R_1, \ldots, R_h$ may contain repetitions and/or the trivial word $R_j = 1 \in F(a_i)$. Those relators nevertheless must *not* be suppressed when constructing the standard complex $K_{\mathcal{P}}$, which hence may contain different 2-cells with the same boundary and/or closed 2-cells that are 2-spheres.

There is no general rule as to whether the R_j are assumed to be (cyclically) reduced or whether R_j may contain adjacent inverse letters a_i, a_i^{-1}. Reducing such “spurs” in the φ_j does not affect the (simple-)homotopy type of $K_{\mathcal{P}}$, see Lemma 2.1 of § 2.1 below; but it may drastically change the embedding behavior of $K_{\mathcal{P}}$ into 4-space, see § 3.2 below.

Amongst the standard complexes are those for canonical dissections of closed 2-manifolds, ([Schu64], III 5.8; [Si92], Chapter 13), but in general more than two local sheets may meet at an e^1. It is nevertheless possible to apply certain moves to finite 2-dimensional CW-complexes which reduce the local complexity by achieving general position; see Ikeda [Ik71], Wright [Wr73], [Wr77] and Remark 1 after the proof of Theorem 3.1 in § 3.1. The result is a compact 2-dimensional polyhedron in which each point has a star of one of the following p.l. types:

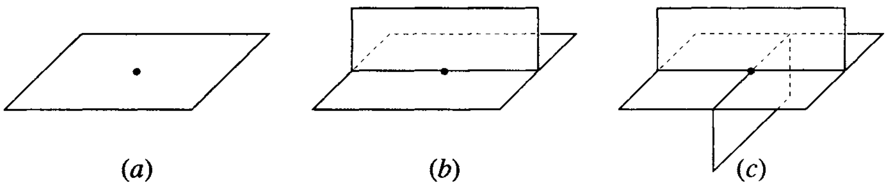

Figure I.8.

Such a polyhedron is called a *closed fake surface*; its *intrinsic 2-skeleton* consists of points of type (a); type (b) resp. (c) defines the *intrinsic 1-resp. 0-skeleton*[6]. As closed 2-manifolds show, these intrinsic skeleta in general don’t dissect a closed fake surface into a cell complex. This additional requirement gives rise to the following definition:

(10) A *special* (or: *standard* polyhedron) is a closed fake surface $|K|$ with the property that the components of the intrinsic skeleta are the cells of a $PL\,CW$-decomposition K (for the given p.l. structure) of $|K|$.

Standard polyhedra are also Reidemeister complexes (Exercise).

In the literature, one sometimes has to deduce the precise meaning of “2-complex” from the context; and if specific names are given, their meaning is not always the same.

[6] Casler [Ca65] and Wright [Wr77] define the intrinsic 1-skeleton to consist of the points of type (b) *or* (c); consequently, these authors don’t introduce the separate notion of the intrinsic 2-skeleton. Compare also the notions of Chapter XI, § 5.1.