1 Introduction

Recall that a *symplectic manifold* is a $2n$-dimensional smooth manifold M together with a closed nondegenerate 2-form ω. A *symplectomorphism* is a diffeomorphism of M which preserves ω and an n-dimensional submanifold $L \subset M$ is called *Lagrangian* if ω vanishes on TL. Such structures arise naturally from Hamiltonian dynamics and geometric optics and they have been studied for many decades. The past ten years have seen a number of important developments and major breakthroughs in symplectic geometry as well as the discovery of new links with other subjects such as dynamical systems, topology, Yang-Mills theory, theoretical physics, and singularity theory.

Many of these new developments have been motivated by Gromov's paper [3] on pseudoholomorphic curves in symplectic geometry. The role pseudoholomorphic curves play in Gromov's work is reminiscent of the role of self-dual Yang-Mills instantons in Donaldson's theory on smooth 4-manifolds. Gromov used pseudoholomorphic curves to prove a number of surprising and hitherto inaccessible results in symplectic geometry. For example he proved that there is no symplectic isotopy moving the unit ball in $\mathbb{R}^{2n}$ through a hole in a hypersurface whose radius is smaller than 1 (*a symplectic camel cannot pass through the eye of a needle*). The paper by McDuff and Traynor below gives a proof of this theorem which is based on Eliashberg's techniques of filling by pseudoholomorphic discs.

Moduli spaces of pseudoholomorphic curves also play an important role in McDuff's work on symplectic 4-manifolds. In her contribution below she proves a uniqueness theorem for symplectic structures on $\mathbb{C}P^2$ with one or two points blown up. This problem is related to the question of connectedness of the space of symplectic embeddings of two disjoint balls into $\mathbb{C}P^2$. The paper by Ciriza deals with the uniqueness of symplectic structures for Kähler manifolds of nonpositive sectional curvature.

Gromov also proved that for every embedded compact Lagrangian submanifold $L \subset \mathbb{R}^{2n}$ there exists a holomorphic disc with boundary on L (a kind of generalization of the Riemann mapping theorem). This can be interpreted as an obstruction to Lagrangian embeddings. Polterovich in his paper proves new such obstructions involving the Maslov class.

Another result by Gromov is his celebrated *squeezing theorem* which asserts that there is no symplectic embedding of the unit ball in $\mathbb{R}^{2n}$ into a cylinder $B^2(r) \times \mathbb{R}^{2n-2}$ of radius less than 1. As a result he proved that the group of symplectomorphisms is closed with respect to the C^0 topology. Hofer interpreted Gromov's squeezing theorem as an example of symplectic invariants which he termed *capacities*. He discovered a number of other capacities, for example the *displacement energy*. In their contribution Hofer and Eliashberg prove a new inequality for the displacement energy and use this to deduce C^1 properties of a hypersurface in $\mathbb{R}^{2n}$ from C^0 information.

This can be viewed as an example of symplectic rigidity.

Another example of symplectic rigidity is Arnold's conjecture about the fixed points of exact symplectomorphisms (time-1-maps of Hamiltonian flows) on compact symplectic manifolds. He conjectured that the number of fixed points of such a symplectomorphism is bounded below by the sum of the Betti numbers. For the torus this was proved by Conley and Zehnder [1] using Morse theory for the symplectic action functional on the loop space. Angenent in his paper uses the symplectic action functional to give a new interpretation of Melnikov's formula for transverse intersections of stable and unstable manifolds in area preserving diffeomorphisms. Zehnder's paper deals with stability problems for symplectomorphisms of $\mathbb{R}^{2n}$. For $n = 1$ this is related to the existence of quasi-periodic solutions which can be established by KAM theory.

An important breakthrough came when Floer combined the variational approach of Conley and Zehnder with Gromov's elliptic techniques and Witten's approach to Morse theory to prove the Arnold conjecture for monotone symplectic manifolds [2]. His work can be summarized as an infinite dimensional version of Morse theory for the symplectic action where the critical points are periodic orbits of Hamiltonian systems and connecting orbits are pseudoholomorphic curves. The resulting invariants are the Floer homology groups. A similar version of Floer homology as an invariant of homology-3-spheres is closely related to Donaldson's theory of smooth 4-manifolds. This amplifies the close relation of pseudo-holomorphic curves in symplectic manifolds with self-dual Yang-Mills equations in 4 dimensions.

The relation between symplectic geometry and gauge theory is fundamental in two of the papers. The moduli space of flat connections over a Riemann surface is a symplectic manifold on which the mapping class group acts by symplectomorphisms. The paper by Dostoglou and Salamon examines the Floer homology groups of these symplectomorphisms. An entirely different relation between contact geometry and gauge theory was discovered by Rumin and this is explained by Pansu in his contribution.

The paper by Donaldson describes new links between *complex-symplectic structures* on 4-dimensional cobordisms (to be thought of as a complexification of the diffeomorphism group of a 3-manifold) and Ashtekhar's formulation of the self-dual Einstein equations.

The papers by Kazarian and Lerman/Montgomery/Sjamaar deal with singularities in symplectic geometry. In the former the singularities arise from geometric optics while the latter deals with symplectic reduction in cases where the quotient is not a manifold.

The paper by Robbin and Salamon explains how the metaplectic representation can be obtained from Feynman path integrals in phase space with general quadratic Hamiltonians. This leads to a simple model of Segal's axioms for topological quantum field theory.

References

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Acknowledgements

I would like to thank the Science and Engineering Research Council and the London Mathematical Society for their generous support of the Conference at Warwick. I also would like to thank all those who helped in running the conference, in particular Shaun Martin and Elaine Shiels.

About this volume

This volume is based on lectures given at a workshop and conference on symplectic geometry at the University of Warwick in August 1990. The area of symplectic geometry has developed rapidly in the past ten years with major new discoveries that were motivated by and have provided new links with many other subjects such as dynamical systems, topology, gauge theory, mathematical physics and singularity theory. The conference brought together a number of leading experts in these interacting areas of mathematics. The contributions to this volume reflect the richness of the subject and include expository papers as well as original research. They will be an essential source for all research mathematicians in symplectic geometry.

Short description

This volume contains expository and research papers by leading experts in symplectic geometry and topology. The contributions reflect the rapid developments in this area in the past ten years and the diversity of the subject. They illuminate the interactions with many other areas such as dynamical systems, topology, gauge theory, mathematical physics and singularity theory.

A Variational Interpretation of Melnikov's Function and Exponentionally Small Separatrix Splitting

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2 February, 1993

§1. Introduction

This note is about the exponentially small separatrix splitting which occurs when one studies the separatrices of maps of "standard type"

$$\Phi(u, v) = (u + \varepsilon, v + \varepsilon f_0(u + \varepsilon v)),$$

where f_0 is an entire function, or when one considers the Poincaré–map associated with the ODE

$$u''(t) = F(t/\varepsilon, u(t)) \tag{1.1}$$

for small values of $\varepsilon > 0$, and nonlinearities F with $F(t+1, u) = F(t, u)$ which are analytic in the u variable. We recall that the Poincaré–map Φ_ε is defined in terms of the first order system

$$u' = v, v' = F(t/\varepsilon, u)$$

which is equivalent to the second order ODE (1.1); Φ_ε sends $(u(0), v(0))$ to $(u(\varepsilon), v(\varepsilon))$, where $(u(t), v(t)), (0 \le t \le \varepsilon)$ is a solution of (1.1). For small

$\varepsilon > 0$ the theory of averaging tells us that we may regard (1.1) as a "small" perturbation of the averaged equation

$$u'' = F_0(u) \tag{1.2}$$

with $F_0(u) = \int_{\mathbf{T}} F(\tau, u) d\tau$.

If the Poincaré map associated with (1.2) has a hyperbolic fixed point with a homoclinic orbit, then one expects the same to be true[1] for the perturbed Poincaré map Φ_ε ($\varepsilon << 1$). One also expects the homoclinic orbit of perturbed map to come from a transverse intersection of the invariant manifolds through the hyperbolic fixed point. Melnikov's method allows one to verify this for smooth perturbations of (1.2) with fixed period, e.g. equations of the form $u'' = F_0(u) + \mu g(t, u)$. However, it has been observed that the method does not apply directly to (1.1). Holmes, Marsden and Scheurle[2] were the first to try to adjust Melnikov's method to the averaging situation. They gave an asymptotic expression for the separatrix splitting if F is of the form $F(\tau, u) = \sin u + \delta\varepsilon^p g(t)$, with g periodic, p sufficiently large, and δ a small parameter. This result was later improved by various authors, the best result to date being due to Delshams, Teresa and Seara[3].

In section 2 we give a variational interpretation of the Melnikov function, and in the subsequent sections show how this interpretation can be adapted to study the homoclinic orbits of (1.1). Like Holmes et.al. we only get an upper bound for the size of the splitting in the most general setting, while we only get transverse homoclinic intersections for a special nonlinearity, $F(\tau, u) = u - {}^3/_2 u^2 + \delta\varepsilon^{10} H'(\tau)$, with $H(\tau)$ periodic, and δ a small parameter. For this particular example the variational approach is an improvement on the results of Holmes et.al. but fails to give the result of Delshams et.al.

One advantage the variational point of view may have over others, is that it can easily be generalized, to find entire solutions of elliptic PDE's such as

$$\Delta u = F_0(u) + \mu g(x, u(x)), \qquad u(\infty) = 0, \tag{1.3}$$

where $g(x, u)$ is periodic in the x-variable; given a nontrivial solution $U(x)$ of the spatially homogeneous equation $\Delta u = F_0(u)$ which vanishes at $x = \infty$,

1 See chapter 4 of [GH83] for a discussion of averaging.

2 [HMS88]

3 See [DTS91] and the references given there.

the analysis in section 2 allows one to find solutions of (1.3) close to some translate $U(x+\vartheta)$ of $U(x)$.

Although there is no obvious Poincaré–map in this situation, one can still show[4] that nondegenerate solutions of (1.3) generate many more solutions of (1.3), much in the same way that a transverse homoclinic point of the Poincaré–map generates an abundance of homoclinic orbits.

The two main examples we have in mind throughout the paper are a forced Duffing equation

$$u'' - F_0(u) = \delta g(t), \tag{1.4}$$

and a "kicked anharmonic oscillator"

$$u''(t) = \varepsilon \sum_{j\in\mathbf{Z}} \delta(t-j\varepsilon)\cdot F_0(u(t)) \tag{1.5}$$

with $F_0(u) = u - {}^3/{}_2u^2$.

In the second example the equation is to be interpreted in the sense of distributions: a solution is a Lipschitz function whose second distributional derivative satisfies (1.5). In fact, solutions will be piecewise linear, and their values $u_j = u(j\varepsilon)$ satisfy the recurrence relation

$$u_{j+1} - 2u_j + u_{j-1} = \varepsilon^2 F_0(u_j). \tag{1.6}$$

One easily verifies that the Poincaré–map Φ_ε is given by the standard type map $(u,v)\mapsto(u+\varepsilon v, v+\varepsilon F_0(u+\varepsilon v))$.

In section 3 we introduce a large class of nonlinearities F which includes both of these examples.

§2. A variational account of the Melnikov function

We assume in this section that $\varepsilon = 1$, and that the nonlinearity F is of the form $F(t,u) = F_0(u) + \mu g(t,u)$, where g is some smooth function, μ is small, and F_0 satisfies

$$F_0(0) = 0, \quad F_0'(0) > 0. \tag{2.1}$$

This last condition implies that the origin is a hyperbolic fixed point for the local flow Ψ_t generated by the system $u' = v, v' = F_0(u)$.

[4] See [A86].

The potential energy associated with F_0 is given by

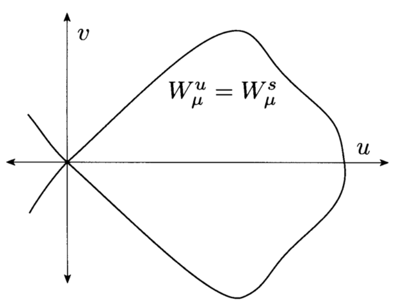
$$V_0(u) = -\int_0^u F_0(\omega)d\omega.$$

We shall assume that $V_0(u) < 0$ for some $u > 0$, and that $V_0'(\alpha) = -F_0(\alpha) < 0$ where α is the smallest positive root of $V_0(\alpha) = 0$. Under this assumption the stable and unstable manifolds W^u, W^s of the origin coincide; they are parametrized by $(U(t), U'(t))$ where U is the unique positive and even solution of

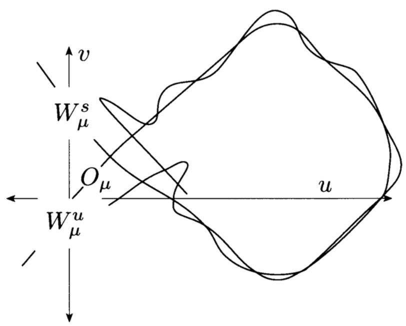
$$U'' = F_0(U), \qquad U(\pm\infty) = 0. \tag{2.2}$$

Consider the Poincaré map Φ_μ of the perturbed system $u' = v, v' = F(\mu, t, u)$. If μ is small Φ_μ will have a hyperbolic fixed point $\mathcal{O}_\mu$ near the origin, whose stable and unstable manifolds we denote by W^s_μ, W^u_μ. Since Φ_μ depends smoothly on μ, the fixed point $\mathcal{O}_\mu$ as well as the W^s_μ, W^u_μ vary smoothly with μ.

For most perturbations $g(t, u)$ the invariant manifolds W^s_μ, W^u_μ will not coincide when $\mu \neq 0$. Melnikov's method was designed to compute the separation between the invariant manifolds for small values of μ, and in particular, to find the transverse intersections in $W^s_\mu \cap W^u_\mu$.



$\mu = 0$



$0 < \mu \ll 1$

It is a commonplace[5] to remark that these transverse intersections are of interest since they are known to be a cause of complicated dynamics of the Poincaré map Φ_μ.

We shall now proceed to describe a variational method which produces a result equivalent to Melnikov's. To begin with we construct a periodic

5 See [Mo73, HG84] and the references given there.

solution which corresponds to the hyperbolic fixed point $\mathcal{O}_\mu$ by applying the implicit function theorem to the map $\mathcal{F} : \mathbf{R} \times C^2(\mathbf{T}) \to C^0(\mathbf{T})$ given by

$$\mathcal{F}(\mu, p) = p'' - F_0(p) - \mu g(t, p).$$

Here $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ and $C^k(\mathbf{T})$ is the space of k times continuously differentiable functions $u(t)$ with $u(t+1) \equiv u(t)$.

We have $\mathcal{F}(0,0) = 0$ while $\mathrm{d}_p\mathcal{F}(0,0)$, the derivative of $\mathcal{F}$ w.r.t. p, is given by $\mathrm{D}^2 - F_0'(0)$; since $F_0'(0) > 0$ the operator $\mathrm{d}_p\mathcal{F}(0,0)$ has a bounded inverse from $C^0(\mathbf{T})$ to $C^2(\mathbf{T})$, and we have a smooth branch of solutions $p(\mu, \cdot) \in C^2(\mathbf{T})$ of $\mathcal{F}(\mu, p) = 0$ with $p(0,t) \equiv 0$. The fixed point $\mathcal{O}_\mu$ is now given by $(p(\mu,0), p'(\mu,0))$.

Homoclinic orbits of Φ_μ correspondend to solutions $u(t)$ of $u'' = F(\mu, t, u)$ which are defined for all $t \in \mathbf{R}$, and which are asymptotic to the small solution $p(\mu,t)$ as $t \to \pm\infty$. To find such solutions we substitute $u(t) = v(t) + p(\mu,t)$ and obtain the following equation for v:

$$v'' = \hat{F}(\mu, t, v(t)), \qquad v(\pm\infty) = 0, \tag{2.3}$$

where

$$\begin{aligned}\hat{F}(\mu,t,v) &= F(\mu,t,p+v) - p'' \\ &= F_0(p+v) - F_0(p) + \mu\,\{g(t,p+v) - g(t,p)\}\end{aligned}$$

with $p = p(\mu,t)$, and $' = \partial/\partial t$. The corresponding potential energy is given by

$$\hat{V}(\mu,t,v) = \int_0^v \hat{F}(\mu,t,\omega)d\omega;$$

it satisfies $|\hat{V}(\mu,t,v)| \le Cv^2$ for small v, and hence the functional

$$\mathcal{A}_\mu(v) = \mathcal{A}(\mu,v) = \int_{\mathbf{R}} \left({}^1\!/_2 v'(t)^2 - \hat{V}(\mu,t,v(t))\right) dt \tag{2.4}$$

is well defined for $v \in H^1(\mathbf{R})$.

2.1. Lemma. *Critical points of $\mathcal{A}_\mu$ are exactly the solutions of (2.3), and hence they correspond to the homoclinic orbits of Φ_μ, i.e. to the intersections of W^s_μ and W^u_μ.*

For small μ a critical point of $\mathcal{A}_\mu$ is nondegenerate if and only if the corresponding intersection of W^s_μ and W^u_μ is transverse.

Proof. The first statement holds since (2.3) is the Euler-Lagrange equation for $\mathcal{A}_\mu$.

Concerning the connection between nondegeneracy and transversality we remark first of all that the Poincaré–maps Φ_μ and $\hat{\Phi}_\mu$, where the latter is derived from $w'' = \hat{F}(\mu, t, w)$, are conjugate. The conjugation is provided by the translation

$$\tau : (u_0, u_0') \mapsto (v_0, v_0') = (u_0 + p(\mu, 0), u_0' + p_t(\mu, 0)).$$

Thus if $P \in W_\mu^u \cap W_\mu^s$, then $\tau(P) \in \hat{W}_\mu^u \cap \hat{W}_\mu^s$, and $\hat{W}_\mu^u$ and $\hat{W}_\mu^s$ intersect transversally at $\tau(P)$ iff W_μ^u and W_μ^s do so at P. We may therefore consider $\hat{W}_\mu^u$ and $\hat{W}_\mu^s$ instead of W_μ^u and W_μ^s.

Let $v \in H^1$ be a critical point of $\mathcal{A}_\mu$. Then $v \in C^\infty$, and $P = (v(0), v'(0))$ is the corresponding intersection of $\hat{W}_\mu^u$ and $\hat{W}_\mu^s$. The second derivative of $\mathcal{A}_\mu$ at v is given by

$$\mathrm{d}^2\mathcal{A}_\mu(v) \cdot (\varphi, \psi) = \langle \mathrm{L}\varphi, \psi \rangle,$$

where $\mathrm{L} : H^1 \to H^{-1}$ is the differential operator

$$\mathrm{L} = -\mathrm{D}^2 + Q(t); \qquad Q(t) =_{\mathrm{def}} \frac{\partial \hat{F}}{\partial u}(\mu, t, v(t)),$$

and where $\langle \varphi, \psi \rangle = \int_{\mathbf{R}} \varphi\psi$.

When μ is small $p(\mu, t)$ is also small, so it follows from

$$\hat{F}_u(\mu, t, v) = F_0'(p(\mu, t) + v) + \mu g_u(t, p(\mu, t) + v),$$

$F_0'(0) > 0$, and $v(\pm\infty) = 0$ that for small μ

$$\liminf_{t \to \pm\infty} Q(t) > 0. \tag{2.5}$$

Hence L is Fredholm with index zero for small μ.

Indeed, $\mathrm{L}_0 = -\mathrm{D}^2 + Q_0(t)$ with $Q_0(t) = \hat{F}_u(\mu, t, 0)$ is invertible, since $\inf Q_0(t) > 0$; $\mathrm{L} - \mathrm{L}_0$ is given by multiplication with $Q(t) - Q_0(t)$, which vanishes at $t = \pm\infty$ and hence is a compact operator from H^1 to H^{-1}; so L is indeed Fredholm.

By definition the critical point v will be nondegenerate iff $\mathrm{L} = \mathrm{d}^2\mathcal{A}_\mu(v)$ is invertible, which, due to L's Fredholmness, will be the case iff L is injective. The nullspace of L consists of those $y \in H^1$ which satisfy

$$y'' = Q(t)y. \tag{2.6}$$