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Introduction

A group Γ with a given system of generators $\{\gamma_i\}_{i \in I}$ carries a unique *maximal* left invariant distance function for which

$$\operatorname{dist}(\gamma_i, id) = \operatorname{dist}(\gamma_i^{-1}, id) = 1, \qquad i \in I.$$

This distance function, called the *word metric* associated to the generating set $\{\gamma_i\} \subset \Gamma$, makes Γ a subject to a geometric scrutiny as any other metric space.

This space may appear boring and uneventful to a geometer's eye since it is discrete and the traditional local (e.g. topological and infinitesimal) machinery does not run in Γ. To regain the geometric perspective one has to change one's position and move the observation point far away from Γ. Then the metric in Γ seen from the distance d becomes the original distance divided by d and for $d \to \infty$ the points in Γ coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps or holes and fills our geometer's heart with joy. For example, an Abelian group Γ with a finite generating set $\{\gamma_i\}$ and the corresponding family of metric, $\operatorname{dist}_{\{\gamma_i\}}/d$, $d > 0$, turns in the limit for $d \to \infty$ into a real linear space L of dimension $n = \operatorname{rank} \Gamma$ with a *Minkowski metric* (also called a Banach norm) whose unit ball around the origin is a convex centrally symmetric polyhedron in L.

Instead of passing to the limit of metric spaces,

$$\lim_{d \to \infty} (\Gamma, \operatorname{dist}/d),$$

(technically speaking, one appeals here to the topology in the set of "all" metric spaces coming along with the *Hausdorff metric*; if the ordinary limit does not exist, one resorts to *ultralimits*, see 2.A), one may remain in the original metric space $(\Gamma, \operatorname{dist}_{\{\gamma_i\}})$ and concentrate on the *asymptotic* properties of Γ which are expressed in terms of distances between variable points in Γ as these distances $\to \infty$.

0.1. Example: the growth function. Let Γ be a discrete metric space and consider the concentric balls of radii d around a chosen point $\gamma_0 \in \Gamma$,

$$B(d) = \{\gamma \in \Gamma \mid \operatorname{dist}(\gamma, \gamma_0) \leq d\}.$$

To make the discussion meaningful, we assume that the balls $B(d)$ are finite (subsets) for all d (which is obviously the case for the word metrics of finitely generated groups) and then we have the growth function of Γ that is

$$N(d) = \operatorname{card} B(d).$$

For small values of d the function $N(d)$ strongly depends on γ_0 and it is oversensitive to perturbations of the metric in Γ. On the other hand, the behaviour of $N(d)$ for large $d \to \infty$ is essentially independent of γ_0 (under mild assumptions on Γ which are satisfied in all examples we are concerned with in this article) and this behaviour is also rather stable under reasonable changes of the metric.

0.1.A. Subexample: growth of an Abelian group. Let Γ be an Abelian group with the word metric corresponding to a finite generating set. Then (this is almost obvious) $N(d)$ has polynomial growth of degree $n = \operatorname{rank}\Gamma$, i.e.

$$A_1 d^n \leq N(d) \leq A_2 d^n + 1, \qquad (*)$$

where A_1 and A_2 are some positive constants depending on the chosen system of generators. It is also not hard to show that there exists a limit

$$A = \lim_{d\to\infty} d^{-n}\, N(d), \qquad (**)$$

which is an improvement over the above inequality $(*)$ for large d. (In fact, the convergence in $(**)$ is quite fast, $A - d^{-n}\, N(d) = O(d^{n-1})$, and it is known to some people in certain quarters when $N(d)$ is actually an honest polynomial in d, compare [Ehr], [Bens], [McM], [Ka-Kho].

0.2. Large-scale equivalence relations between metric spaces. Our "asymptotic" attitude obliges every such equivalence relation to be strong enough to make every *bounded* space X equivalent to a single point (or, at least to an arbitrarily small space). Recall that a metric space X is called *bounded* if

$$\operatorname{Diam} X \underset{\text{def}}{=} \sup_{x_1,x_2} \operatorname{dist}(x_1, x_2) < \infty.$$

Here is the weakest relation of this sort used in geometry:

0.2.A. Hausdorff equivalence between metric spaces. Write

$$X \underset{\text{Hau}}{\sim} Y,$$

where X and Y are metric spaces, if there exists a metric on the disjoint union Z of X and Y, such that dist_Z on X equals the original metric dist_X on X and similarly $\operatorname{dist}_Z|Y = \operatorname{dist}_Y$, such that the distance functions

$$\delta(x) = \operatorname{dist}_Z(x, Y) \underset{\text{def}}{=} \inf_{y\in Y} \operatorname{dist}_Z(x, y)$$

and

$$\delta(y) = \text{dist}_Z(y, X)$$

are *bounded*, i.e.

$$\sup_{x \in X} \delta(x) < \infty \qquad \text{and} \qquad \sup_{y \in Y} \delta(y) < \infty.$$

Recall that the maximum of the above two suprema is called the *Hausdorff distance* (between subsets X and Y in Z) and the *infimum* of these distances over all metrics on Z which restrict to dist_X on $X \subset Z$ and dist_Y on $Y \subset Z$ is called the (abstract) *Hausdorff distance between metric spaces* X and Y. Thus the relation $X \underset{\text{Hau}}{\sim} Y$ expresses the finiteness of $\text{dist}_{\text{Hau}}(X, Y)$. (Our discussion on the limit of spaces at the beginning of this introduction refers to the convergence of unbounded spaces, $X_i \to X$, $i = 1, 2, \dots$, with respect to the Hausdorff distance between appropriately chosen *bounded* subsets $B_i \subset X_i$ and $B_i' \subset X$. Then the Hausdorff convergence $X_i \to X$, does not preclude the infinite Hausdorff distance between X and every X_i, $i = 1, 2, \dots,$. This is similar to the uniform convergence of functions on bounded or compact subsets of a fixed infinite space, such as $\mathbb{R}^n$, for instance.)

0.2.A$_1$. *Example.* Let Γ be a free Abelian group of rank n and $\{\gamma_1, \dots, \gamma_n\}$ be a (free) system of generators. Then Γ with the corresponding word metric is $\underset{\text{Hau}}{\sim}$ to the n-dimensional Euclidean space $\mathbb{R}^n$ with the so-called ℓ_1-metric

$$\text{dist}(x, y) = \sum_{i=1}^{n} |x_i - y_i|.$$

In fact, the homomorphism $\Gamma \to \mathbb{R}^n$ extending

$$\gamma_1 \mapsto (1, 0, \dots, 0),\ \gamma_2 \mapsto (0, 1, 0, \dots, 0),\ \dots,$$

is an isometry and every point of $\mathbb{R}^n$ lies at most distance one from the image of Γ.

0.2.A$_2$. Long-range connectedness. Here is the simplest instance of redefining a standard topological notion in the large-scale terms. A metric space X is called *long-range* (or large-scale) *connected* if there exists a constant $d > 0$ such that every two points x and y in X can be joined by a finite chain of points

$$x_0 = x,\ x_1,\ x_2,\ \dots, x_n = y,$$

such that

$$\text{dist}(x_i, x_{i-1}) \le d, \quad i = 1, \dots, n.$$

It is clear, that the long range connectedness in invariant under $\underset{\text{Hau}}{\sim}$. In fact, X is *l.r. connected* if and only if it is $\underset{\text{Hau}}{\sim}$ to a path connected space. (*Idea of the proof:* add to X the edges between all pairs of points with mutual distances $\leq d$ and extend the metric from X to the resulting space $X_d \supset X$ of paths.

Example. If $X = (\Gamma,$ word metric) then X_1 equals the Cayley graph of Γ which, as we know, is always connected.)

0.2.A$'_2$. L.r. connectedness at ∞. The idea of l.r. connectedness becomes interesting in the group theoretic context when it applies not to a group Γ directly, but to some auxiliary space or a sequence of spaces. An instance of that is *l.r. connectedness at infinity* defined as follows.

A metric space X is called *l.r. disconnected* at infinity if for every $d > 0$ there exist two subsets X_1 and X_2 in X such that

(i) $\text{dist}(X_1, X_2) \geq d$ which means by the definition of this dist between subsets that

$$\text{dist}(x_1, x_2) \geq d \quad \text{for all} \quad x_1 \in X_1, \quad \text{and} \quad x_2 \in X_2.$$

(ii) X_1 and X_2 cover almost all X, i.e. the complement $X - (X_1 \cup X_2)$ is bounded.

Then X is called *l.r. connected at* ∞ if for some d the above X_1, X_2 do not exist.

Similarly, using k different X_i instead of two, one defines the *number* of l.r. connected components at ∞ which agrees with the usual notion of the *ends* of groups.

A remark relevant to our discussion is the invariance of the number of ends (i.e. l.r. components at ∞) under the Hausdorff equivalence.

0.2.B. Terminology: "asymptotic", "long-range", "large-scale". These expressions are used interchangeably and the choice of a particular one depends on what kind of associations we want to carry along with a formal argument. Thus "asymptotic" awakens an analyst in our minds, "large scale" shifts the discussion into a more geometric vein and "long range" appeals to whatever is left in us of a physicist.

0.2.C. Lipschitz equivalence and quasi-isometry. Two metrics on the same space, say dist_1 and dist_2, are called (Lipschitz) *equivalent* if the ratios $\text{dist}_1 / \text{dist}_2$ and $\text{dist}_2 / \text{dist}_1$ are *bounded* when they are considered as functions on the Cartesian square of the space minus the diagonal. Then two different metric spaces X_1 and X_2 are called *(bi-)Lipschitz equivalent* if there exists a bijection $X_1 \to X_2$ which brings the metric from X_1 to a metric on X_2 which is equivalent to the original metric on X_2.

Example. If dist_1 and dist_2 are word metrics on Γ corresponding to two finite generating sets then they are (obviously) equivalent. Consequently, isomorphic finitely generated groups are $\underset{\mathrm{Lip}}{\sim}$ (this is an abbreviation of "Lipschitz equivalent") for their respective word metrics.

Remark. One can alternatively define the Lipschitz equivalence as an isomorphism in the *category* of *metric spaces* and *Lipschitz map* where a map $f : X_1 \to X_2$ is called Lipschitz if there exists a (Lipschitz) constant $\lambda \geq 0$, such that

$$\mathrm{dist}(f(x), f(y)) \leq \lambda\ \mathrm{dist}\,(x, y) \quad \text{for all } x,\, y \in X_1.$$

Notice, that every homomorphism between finitely generated groups is Lipschitz.

Now we use both relation $\underset{\mathrm{Hau}}{\sim}$ and $\underset{\mathrm{Lip}}{\sim}$ and generate with them what is called the *quasi-isometry* equivalence between metric spaces X and Y. In fact, X and Y are quasi-isometric if and only if there exist X' and Y', such that

$$X \underset{\mathrm{Hau}}{\sim} X' \underset{\mathrm{Lip}}{\sim} Y' \underset{\mathrm{Hau}}{\sim} Y.$$

0.2.C$_1$. *Basic example.* Let X be a Riemannian manifold and let Γ be a finitely generated group *properly* and *isometrically* acting on X. (An action of a discrete group is *proper* if for every compact subset $B \subset X$ the intersection $B \cap \gamma(B)$ is empty for almost all, i.e. for all but finitely many $\gamma \in \Gamma$.) Next, a proper action is called *cocompact* if the quotient space X/Γ is compact. This is equivalent (for the proper actions) to the existence of a compact subset $B \subset X$ whose Γ-translates cover all of X, i.e. $\Gamma B = X$.

The following obvious proposition-example constitutes the major link between the asymptotic group theory and the large-scale Riemannian geometry.

If the action of Γ on X is proper and cocompact then Γ is quasi-isometric to X.

(Here and in future, Γ is given the word metric associated to some *finite* generating set.)

Corollary. *There exist quasi-isometric groups Γ_1 and Γ_2 which are not commensurable.* (Recall that Γ_1 and Γ_2 are *commensurable* if there exist subgroups of finite index, $\Gamma_1' \subset \Gamma_1$ and $\Gamma_2' \subset \Gamma_2$ such that Γ_1' is isomorphic to Γ_2'.)

For example, the product of two hyperbolic planes, $X = H^2 \times H^2$, admits an irreducible cocompact proper action of a discrete group Γ, where "irreducible" means that the induced action of Γ (or rather of the subgroup $\Gamma' \subset \Gamma$ of index ≤ 2 which does not interchange the Cartesian components of $H^2 \times H^2$) on each H^2 is non-proper. Such a Γ is quasi-isometric to the product $\Gamma_1 \times \Gamma_2$ of two surface groups (as $\Gamma_1 \times \Gamma_2$ obviously acts on $H^2 \times H^2$) but one can easily show that Γ is not commensurable to $\Gamma_1 \times \Gamma_2$. (The only truly non-trivial point in the above discussion is the existence of an irreducible Γ. This is constructed by arithmetic means, see [Gr-Pa] for an elementary discussion on the matter.)

0.2.C$_2$. Let us indicate a non-Riemannian version of the above example. Take an arbitrary locally compact group G and consider two discrete subgroups Γ_1 and Γ_2 in G. *Then, if Γ_1 and Γ_2 are finitely generated and cocompact in G then they are quasi-isometric.* Instead of giving a proof (which is trivial anyway) we indicate a further generalization which is motivated by the following features of our picture

(i) The left action of Γ_1 on G commutes with the right action of Γ_2;
(ii) both actions are cocompact on G.

Now we state the following

0.2.C$'_2$. Topological criterion for quasi-isometry. *Two finitely generated groups Γ_1 and Γ_2 are quasi-isometric if and only if there exist proper actions of Γ_1 and Γ_2 on some locally compact topological space X such that*

(i) *the actions commute;*
(ii) *both actions are cocompact.*

Idea of the proof. We only indicate here how to produce an X starting from a quasi-isometry between Γ_1 and Γ_2. To simplify the matter we assume a Lipschitz equivalence rather than a quasi-isometry which is given by a bi-Lipschitz bijection $f : \Gamma_1 \to \Gamma_2$. Then we consider the space F of *all* maps $\Gamma_1 \to \Gamma_2$ with the pointwise convergence (topologically, this is a countable union of Cantor sets) and observe that the natural actions of Γ_1 and Γ_2 on F are proper and they commute. Then we take the closure X of the $(\Gamma_1 \times \Gamma_2)$-orbit of our $f \in F$ and leave it to the reader to check that the actions of Γ_1 and Γ_2 on X are co-compact.

0.2.D. Why Lipschitz? Let us try to relax further our equivalences. Say that two metrics dist_1 and dist_2 on X are *uniformly equivalent on the large-scale (or l.s.u. equivalent)* if there exists a real function $\lambda(d)$, $d > 0$, such that

$$\text{dist}_1(x,y) \leq \lambda(\text{dist}_2(x,y)) \quad \text{for all } x \text{ and } y \text{ in } X$$

and conversely,

$$\text{dist}_2 \leq \lambda(\text{dist}_1).$$

Then one defines the *l.s.u. equivalence* between metric spaces X and Y by mixing the above with the Hausdorff equivalence. This may appear significantly more general than quasi-isometry but it is not quite so because of the following trivial

Lemma. *If the spaces X and Y are quasi-geodesic (see the definition below) then l.s.u. equivalence between X and Y is the same thing as quasi-isometry.*

Definition. A metric space X is called *quasi-geodesic* if there exist positive constants d and λ, such that for every two points x and y in X there exists a finite chain of points in X,

$$x_0 = x,\ x_2, \dots, x_n = y,$$

such that

$$\text{dist}\,(x_i, x_{i-1}) \leq d, \quad i = 1, \dots, n, \tag{$*$}$$

and

$$\sum_{i=1}^{n} \text{dist}\,(x_i, x_{i-1}) \leq \lambda\ \text{dist}\,(x, y). \tag{$\substack{*\\*}$}$$

Examples. (a) Every group Γ with a word metric is (obviously) quasi-geodesic. In fact it is almost geodesic as one can satisfy $(*)$ and $(\substack{*\\*})$ with $d = 1$ and $\lambda = 1$. (For truly geodesic one asks for an arbitrarily small $d > 0$ in $(*)$.)

(b) Let X be a connected Riemannian manifold. Then it is quasi-geodesic almost by definition as $\text{dist}(x, y)$ appears as the infimum of the lengths of paths in X between x and y. If X is complete as a metric space, then X is truly geodesic as the above infimum is actually achieved by some curve between x and y. (Notice, that this does not exclude manifolds with boundaries which are metrically complete but are not complete in a certain more technical sense.)

(c) Let $\Gamma_1 \subset \Gamma_2$ be a finitely generated subgroup in a finitely generated group Γ. Then the word metric dist_2 restricted to Γ_1 is not, in general, quasi-geodesic in Γ_1. The simplest instance of that is seen in the nilpotent group $\Gamma_2 = \langle a, b, c \mid [a, b] = c,\ [a, c] = [b, c] = 1 \rangle$ for $\Gamma_1 = \mathbb{Z}$ generated by the (central) element c. Here one immediately sees that the commutator $[a^n, b^n]$ lies in Γ_1 and is equal to c^{n^2}. Thus $\text{dist}_2 |\Gamma_1 \underset{\text{Lip}}{\sim} (\text{dist}_1)^{\frac{1}{2}}$, and so dist_1 and dist_2 are uniformly equivalent on Γ_1 but by no means Lipschitz equivalent.

0.3. From groups to spaces. Take a finitely generated group Γ and let dist be a word metric. Now we try to forget the structure of the group in Γ and look on (Γ, dist) as on a metric space. (Forgetting the structure is not quite complete at this stage as Γ appears as a subgroup in the full isometry group Iso (Γ, dist); moreover, Γ = Iso(Γ, dist) in most cases.) Furthermore, as we are interested in the large-scale geometry of (Γ, dist) we want our analysis of Γ to be stable under quasi-isometrics. In other words our (geo)metric invariants should remain unchanged if we pass to a metric space (Γ', dist$'$) quasi-isometric to (Γ, dist). Now it is not at all easy to recognize Γ by looking at Γ', yet a variety of characteristics of Γ can be reconstructed in terms of Γ'! These are precisely the asymptotic (or large scale) invariants we are after. In fact, there are certain cases (e.g. $\Gamma = \mathbb{Z}^n$) where one can recapture the group Γ itself up to commensurability.

Given a discrete metric space Γ, one can make it more palatable by adding some meat to Γ in the form of edges and higher dimensional simplices with vertices in Γ, without changing the quasi-isometry type. For example, if Γ is finitely presented, then there is a finite 2-dimensional polyhedron P with $\pi_1(P) = \Gamma$ and the universal covering $\widetilde{P}$ gives us a nice tasty thickening of Γ as $\widetilde{P}$ is connected and simply connected. There is not, in general, any distinguished metric on $\widetilde{P}$ quasi-isometric to Γ, but there is a reasonable class of such metrics which are invariant under the deck transformation group Γ. A geometrically oriented reader may prefer another version of this construction where instead of P one takes a compact Riemannian manifold V (possibly with a boundary) having the fundamental group $\pi_1(V) = \Gamma$ and then passes to the universal cover $\widetilde{V}$ with the induced Riemannian metric. This is a geodesic metric space which is connected and simply connected and where Γ acts properly and cocompactly. So again $\widetilde{V}$ is quasi-isometric to Γ. Thus the large-scale (or asymptotic) geometry of a finitely presented group embeds into a more general theory, that is the quasi-isometric geometry of non-compact Riemannian manifolds with no group acting anywhere.

Here one may start to feel rather uncomfortable by realizing how much structure has been lost as one passed from Γ to the quasi-isometry class of (Γ, word metric). Indeed one barters here a rigid crystalline beauty of a group for a soft and flabby chunk of geometry where all measurements have built-in errors. But something amazing and unexpected happens here as was discovered by Mostow in 1968: the quasi-isometric (or large-scale) geometry turns out by far more rich and powerful than appears at first sight. In fact, one believes nowadays that most essential invariants of an infinite group Γ are quasi-isometry invariant. Well, even so, why should we go through all the pains of reconstructing the group structure from geometry if nobody forces us to leave the pure group theoretic world in the first place? Here are several reasons to do so.

I. The group theoretic structure appears too rigid and limits one to formal combinatorial and algebraic manipulations with no room for transcendental methods (i.e. the analysis of infinity). This is similar to the elementary theory of metric spaces where the only admissible maps are isometries. It is fruitful to include into the category more morphisms, such as Lipschitz maps, continuous maps, measurable maps etc., thus bringing analysis into play.

II. Even in purely group theoretic questions the geometric *language* may tremendously clarify the picture. For example, from a geometer's (even a topologist's) viewpoint the *free subgroup theorem* ("a subgroup of a free group is free") appears as a painful way of expressing (in a special case) the obvious feature of covering maps $\widetilde{Y} \to Y$,

$$\dim Y = 1 \quad \Rightarrow \quad \dim \widetilde{Y} = 1.$$

(If you have ever tried and failed to drag yourself through the notational rigours of an algebraic proof you must share my relief at the realization that the difficulty there stemmed not from mathematics but from a non-adequate language. I still feel thankful to Dima Kazhdan who explained the matter to me many years ago.) Similar linguistic aberrations can be observed (at least by a geometer) in all corners of the traditional geometric group theory, such as the theory of free products (with and without amalgamations), small cancellation theory etc. (The adherence to the combinatorial language comes from an instinctive mistrust most algebraists feel toward geometry which they regard as "non-rigorous".)

II$'$. ***Example: Hyper-Euclidean groups.*** Here is an instance of a useful notion which naturally pops up in the geometric setting and which would become a major nuisance once one committed oneself to a purely algebraic language.

Definition. A group Γ is called *hyper-Euclidean in dimension n* if it admits a proper isometric action on a connected oriented n-dimensional Riemannian manifold X without boundary which admits a *proper Lipschitz* map $f : X \to \mathbb{R}^n$ of *degree one.* It is sometimes desirable to vary this definition

(a) by requesting the action to be cocompact,
(b) by allowing the action to be quasi-isometric,
(c) by admitting maps f of degree ≥ 1,
(d) by insisting that X should be contractible.

(The hyper-Euclidean conception appears in geometry and topology in the study of positive scalar curvature, see [Gr-Law] and the Novikov higher signature conjecture, see [Fa-Hs] and [C-G-M].)

III. The geometric language brings along a variety of concepts, constructions and ideas unimaginable in the world of pure algebra (such as the above "hyper-Euclidean"). Thus, geometry suggests an impressive number of potentially useful asymptotic invariants of groups about which one may ask the following standard questions,

(A) When and how can one compute such an invariant for a given group? (E.g. how to decide if a given group is hyper-Euclidean.)

(B) What are the relations between different invariants?

(C) Which values of an invariant can be realized by some group Γ? (E.g. when does a given function $f(d)$ appear as the growth function of some finitely generated group Γ? Compare 0.1.)

(D) How large is the class of groups with a given value of an invariant? (E.g. is every group (of finite cohomological dimension) hyper-Euclidean in dimension n for a given n?)

IV. When we go from groups to spaces we mentally change the class of essential examples. The most important manifolds studied by geometers are symmetric and locally symmetric spaces (of finite and infinite dimension) and other homogeneous spaces. Besides being remarkably attractive objects in their own right these spaces may serve as measuring rods for the study of more general spaces and groups. A typical instance of that is the above definition of "hyper-Euclidean" where a general manifold is compared in a certain way with $\mathbb{R}^n$.

V. The last but not the least argument in favour of geometry is applicability of geometric ideas (and very rarely of techniques) to the solution of some group theoretic problems. Unfortunately, this is an exception rather than the rule but the situation will probably change with the development of the field.

(I do not know how convincing the above evidence truly is. After all, the actual reason why one approaches a problem from a geometric angle is because one's mind is bent this way. No amount of rationalization can conceal the truth.)

0.4. About this paper. Our purpose here is to demonstrate the efficiency of the geometric language for defining invariants and isolating interesting properties of groups. In many cases we just specialize the standard notions of the asymptotic geometry to groups in order to make them known to the group theorists. We do not attempt a serious study of our invariants and leave the standard questions wide open. On some occasions we treat simple examples lying immediately on the surface. Often we speculate on the possible outcome of the game only not to lose reader's attention, even when we have no inkling of a viable approach to the solution. Thus the readers of this paper should not expect new theorems (not even half proved ones), but they may come across some amusing problems.

Remarks on the language. We develop many of our notions in the geometrically friendly surroundings of Riemannian manifolds and similar spaces.

This immediately applies to groups in so far as the quasi-isometry invariance of the concepts in question is insured. Namely, in order to attribute some geometric property Pr to a group Γ, we just require Pr for some (and thus every) manifold X quasi-isometric to Γ, where, in addition, we may impose some specific condition on X (e.g. being simply connected, contractible etc) if this is needed for the introduction of Pr. On the other hand if we do not want to bother with the quasi-isometry invariance we have to make our choice: either we insist Pr is satisfied for *all* X (with some specified conditions) quasi-isometric to Γ (sometimes we must insist on a proper isometric action of Γ on X) or we only require the existence of *some* X quasi-isometric to Γ which has Pr. Of course, when a quasi-isometry invariance of some property is unknown it adds a problem to our list.

0.5. Random historical remarks. The first distinctively asymptotic ideas in geometric group theory appeared in the mid-fifties in the papers by Efremovic [Ef], Folner [Fo] and Ŝvarc [Ŝv]. Folner gave a geometric criterion for *amenability* of a finitely generated group Γ. The notion of amenability comes from ergodic theory where a group Γ (which may be infinitely generated) is called *amenable* if every continuous action of Γ on a compact space has an invariant measure.

0.5.A. Folner Criterion. Γ is amenable if and only if there exists an exhaustion of Γ by finite (Folner) subsets $F_1 \subset F_2 \subset \ldots \subset F_i \subset \ldots \Gamma$, such that for every $d > 0$ the *d-boundary* $\partial_d F_i$ (defined below) of F_i has asymptotically a smaller number of elements than F_i,

$$\limsup_{i \to \infty.} \operatorname{card}(\partial_d F_i)/\operatorname{card} F_i = 0$$

0.5.A$_1$. *Definition.* The d-boundary of a subset F in a metric space Γ consists of the points $x \in F$ whose distance to the complement $\Gamma - F$ does not exceed d. (An alternative definition which is as good for the present purpose is where $\partial_d \Gamma$ consists of the points in $\Gamma - F$ within distance $\leq d$ from F.)

0.5.A$'_1$. *Example.* Folner's criterion immediately shows that every finitely generated Abelian group is amenable. On the other hand the standard example of a non-amenable group is the free group $\mathbb{F}_2$ on two generators. Some people naively believed for some time that every finitely generated non-amenable group should contain a copy of $\mathbb{F}_2$ but to day there are counterexamples which are infinitely presented (see [Ols]). One still has no construction of a finitely presented non-amenable group containing no $\mathbb{F}_2$.

It is useful to reformulate the Folner criterion with the emphasis on non-amenability.

0.5.A$_2$. Isoperimetric form of Folner criterion. A group Γ is non-amenable if and only if there exist positive constants d and C, such that every finite subset $F \subset \Gamma$ satisfies

$$\operatorname{card} F \leq C \operatorname{card} \partial_d F \qquad (*)$$

This inequality immediately brings to one's mind the classical linear isoperimetric inequality for bounded domains Ω in the hyperbolic space H^n,

$$\operatorname{Vol}_n \Omega \leq \operatorname{const} \operatorname{Vol}_{n-1} \partial\Omega. \qquad ({}^{*}_{*})$$

In fact, the similarity between $(*)$ and $({}^{*}_{*})$ can be made precise as these inequalities are equivalent for quasi-isometric spaces satisfying the following *bounded geometry* conditions.

0.5.A$_3$. ***Definitions.***

(a) A discrete metric space Γ is said to be *uniformly quasi-locally bounded* (u.q.-l.b.) if there exists a function $N(d)$, $d \geq 0$, such that every ball $B \subset \Gamma$ of radius d has

$$\operatorname{card} B \leq N(d).$$

(b) A Riemannian manifold X has *locally bounded geometry* (l.b.g.) if there exist positive constants ε and λ such that every ε-ball in X is λ-bi-Lipschitz equivalent to the ε-ball $B_0 \subset \mathbb{R}^n$. (This means the existence of a bi-Lipschitz map $B \to B_0$ with the implied constant λ, compare 0.2.C.)

0.5.A$_4$. *Example.* Every finitely generated group Γ is u.q.-l.b. Every Riemannian manifold X without boundary whose full isometry group is cocompact on X has l.b.g. (If X has a boundary the definition needs a minor adjustment.)

0.5.A$_5$. ***Proposition.*** *Let Γ be a discrete* u.q.-l.b. *space and X a Riemannian manifold having* l.b.g. *If X is quasi-isometric to Γ then the inequality $(*)$ for Γ (i.e. for all finite subsets $F \subset \Gamma$) implies $({}^{*}_{*})$ for X (i.e. for all bounded domains $\Omega \subset X$) where the constant in $({}^{*}_{*})$ depends on C in $(*)$ as well as on the implied quasi-isometry. Conversely, $({}^{*}_{*})$ for X implies $(*)$ for Γ.*

The proof appears obvious to a geometrically oriented mind and nowadays even the hard core group theorists are beginning to agree with this view.

0.5.A$_6$. ***Corollary.*** *Let a discrete group Γ admit a proper cocompact action on X. Then Γ is non-amenable if and only if X satisfies the (linear isoperimetric) inequality $({}^{*}_{*})$.*

This applies, in particular, to the universal covering X of a compact manifold V with $\pi_1(V) = \Gamma$.

0.5.B. Also in the fifties Efremovich [Ef] observed that the growth-rate of the volume of the balls in the universal covering X of V, i.e.

$$\mathrm{Vol}\, B(d) \quad \text{for}\, d \to \infty,$$

depends only on the fundamental group Γ of V but not on the particular choice of V. In fact he pointed out (now it looks totally obvious) that $\mathrm{Vol}\, B(d)$ for $d \to \infty$ grows essentially with the same rate as the corresponding function $N_\Gamma(d)$ defined in 0.1,

$$N_\Gamma(d) = \mathrm{card}\, B_\Gamma(d)$$

for the balls $B_\Gamma(d) \subset \Gamma$.

The ideas of the growth of balls, Folner sets and sets of conjugacy classes in groups (especially in fundamental groups of manifolds of negative curvature, see [Mar]$_1$ [Mar]$_2$) were quite popular in the sixties among ergodic theorists in Moskow and Leningrad. (Much of these ideas I learned at the time from A. Vershik, D. Kazhdan and G. Margulis.) Then the geometers took a part in the story and related the growth to curvature. The first results here for non-negative curvature are due to A. Ŝvarc [Ŝv]. Similar results were obtained independently by J. Milnor (see [Mil]) who stated the following

0.5.B$_1$. *Conjecture.* The growth function $N_\Gamma(d)$ of a finitely generated group Γ is either *polynomial* (i.e. $N(d) \leq 1 + Cd^n$ for some positive C and n) or *exponential*, which means

$$N(d) \geq A^d \quad \text{for some}\, A > 1.$$

This conjecture is known to be true for linear groups (i.e. subgroups of GL_N) by the work of Tits who proved the following

0.5.B$_2$. Freedom theorem (see [Tit]$_1$). *Every finitely generated linear group* Γ *is either virtually solvable* (i.e. contains a solvable subgroup of finite index) *or contains a copy of* $\mathbb{F}_2$*, the free group on two generators.* This implies the conjecture, for the groups $\Gamma \supset \mathbb{F}_2$ obviously have exponential growth; furthermore, the virtually solvable groups Γ have $N_\Gamma(d)$ exponential unless they are virtually nilpotent. The latter are known to have polynomial growth and are, in fact, characterized by this property, see [Tit]$_2$ and references therein.

0.5.B_3. Milnor's conjecture is still open for *finitely presented* groups but recently Grigorchuk found a remarkable class of finitely generated infinitely presented groups of *intermediate* growth where $N(d)$ behaves as A^{d^α}, $0 < \alpha < 1$. (Grigorchuk's groups Γ act on an infinite regular tree fixing a vertex and therefore are residually finite without being linear. The essential feature of Γ responsible for the intermediate growth is the existence of mutually isomorphic subgroups $H \subset \Gamma$ and H' is the Cartesian product $\Gamma \times \Gamma \times \Gamma \times \Gamma \times \Gamma \times \Gamma \times \Gamma \times \Gamma$. See [Gri] for a comprehensive survey of the growth theory.) The current version of the growth conjecture due to Grigorchuk reads

There exists $\alpha > 0$ (possibly $\alpha = \frac{1}{2} - \varepsilon$) such that either $N_\Gamma(d)$ grows faster than A^{d^α} or Γ has polynomial growth (and, hence, is virtually nilpotent).

0.5.B_4. There is a simple link between growth and amenability.

If Γ is non-amenable then it has exponential growth.

This immediately follows by applying the (linear isoperimetric) inequality $(*)$ to the concentric balls $B(d) \subset \Gamma$.

Thus Grigorchuk's examples provide a new class of amenable groups. Prior to his work all known amenable groups were obtained from finite and Abelian groups (which are easily seen to be amenable) by the following three operations.

1. *Extensions:* Here one uses the fact that if in the exact sequence $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ the groups Γ_1 and Γ_3 are amenable, then so is Γ_2.
2. *Infinite unions:* If Γ is a union of an increasing family of amenable subgroups then Γ is amenable.
3. *Taking subgroups and factor groups:* Every subgroup of an amenable group is amenable and so is every factor group.

Notice that in the course of such a construction one may have intermediate groups infinitely generated even if the final result is f.g., as was pointed out, I believe, by H. Bass. Also recall that Grigorchuk's groups are *not* finitely presented and one has still no ways to produce finitely presented amenable groups apart from 1, 2 and 3.

0.5.C. The main source of infinite groups in differential geometry is provided by manifolds of non-positive sectional curvature, $K \leq 0$. One of the first asymptotic results here is the following result by A. Avez (see [Av] and §6).

0.5.C_1. Non-amenability theorem. *Let V be a compact manifold without boundary and $K(V) \leq 0$. Then the fundamental group of V is non-amenable unless V is flat (and then $\pi_1(V)$ is virtually Abelian).*

The proof suggested by Avez is based on the following

0.5.C$'_1$. Non-amenability criterion. *Let X be an n-dimensional Riemannian manifold which admits a vector field Z with the following two properties*

(i) *the length of Z is uniformly bounded*

$$\sup_{x \in X} \|Z(x)\| < \infty,$$

(ii) *the divergence of Z is strictly positive,*

$$\inf_{x \in X} \operatorname{div} Z(x) > 0.$$

Then every bounded domain Ω in X with a smooth boundary satisfies

$$\mathrm{Vol}_n \Omega \leq \mathrm{const}\, \mathrm{Vol}_{n-1} \partial \Omega. \qquad (+)$$

Furthermore, the conclusion remains valid if we replace (ii) *by the following weaker condition* (ii)$_0$ *and additionally assume that X has locally bounded geometry* (see 0.5.A$_3$).

(ii)$_0$ div $Z(x) \geq 0$ *for all $x \in X$ and there exist positive numbers d and ε such that for every ball $B \subset X$ of radius d the integrated divergence of Z over B is at least ε,*

$$\int_B \operatorname{div} Z(x)\, dx \geq \varepsilon.$$

Idea of the proof. Integrate div Z over Ω and apply Stokes' theorem.

Avez applies this criterion to the gradients Z of *horofunctions* in the universal covering X of V. Recall that a horofunction $h : X \to \mathbb{R}$ is a limit of a sequence of additively normalized distance functions $h_i(x) = \mathrm{dist}(x, x_i) - c_i$, where $x_i \in X$ is a sequence of points going to infinity and c_i is a sequence of constants. If $K \leq 0$ then horofunctions h (as well as distance functions) are (known to be) convex and so div grad $h \geq 0$. In general, the strict inequality

$$\operatorname{div} \operatorname{grad} h \geq \varepsilon > 0$$

needs strictly negative curvature,

$$K(X) \leq \kappa < 0,$$

but in the case where X covers a compact non-flat manifold V Avez produces a horofunction h whose gradient satisfies (ii)$_0$.