
Contents

Preface	<i>page xi</i>
1 The equations of motion	1
1.1 Introduction	1
1.2 Euler's equations for an incompressible fluid	1
1.3 Energy, body forces, vorticity, and enstrophy	7
1.4 Viscosity, the stress tensor, and the Navier-Stokes equations	12
1.5 Thermal convection and the Boussinesq equations	18
1.6 References and further reading	21
Exercises	22
2 Dimensionless parameters and stability	23
2.1 Dimensionless parameters	23
2.2 Linear and nonlinear stability, differential inequalities	29
2.3 References and further reading	38
Exercises	38
3 Turbulence	40
3.1 Introduction	40
3.2 Statistical turbulence theory and the closure problem	40
3.3 Spectra, Kolmogorov's scaling theory, and turbulent length scales	49
3.4 References and further reading	59
Exercises	60
4 Degrees of freedom, dynamical systems, and attractors	61
4.1 Introduction	61
4.2 Dynamical systems, attractors, and their dimension	62

viii	<i>Contents</i>	
4.3	The Lorenz system	74
4.4	References and further reading	86
	Exercises	87
5	On the existence, uniqueness, and regularity of solutions	88
5.1	Introduction	88
5.2	Existence and uniqueness for ODEs	89
5.3	Galerkin approximations and weak solutions of the Navier-Stokes equations	96
5.4	Uniqueness and the regularity problem	104
5.5	References and further reading	113
	Exercises	113
6	Ladder results for the Navier-Stokes equations	114
6.1	Introduction	114
6.2	The Navier-Stokes ladder theorem	117
6.3	A natural definition of a length scale	125
6.4	The dynamical wavenumbers $\kappa_{N,r}$	127
6.5	Estimates for the Navier-Stokes equations	128
	6.5.1 Estimates for F_0	129
	6.5.2 Estimates for $\langle F_1 \rangle$ and $\langle \kappa_{1,0}^2 \rangle$	130
	6.5.3 Estimates for $\overline{\lim}_{t \rightarrow \infty} F_1$, $\langle F_2 \rangle$, and $\langle \kappa_{2,1}^2 \rangle$	131
6.6	A ladder for the thermal convection equations	132
6.7	References and further reading	134
	Exercises	134
7	Regularity and length scales for the 2d and 3d Navier-Stokes equations	137
7.1	Introduction	137
7.2	A global attractor and length scales in the 2d case	138
	7.2.1 A global attractor	139
	7.2.2 Length scales in the 2d Navier-Stokes equations	139
7.3	3d Navier-Stokes regularity?	144
	7.3.1 Problems with 3d Navier-Stokes regularity	144
	7.3.2 A Bound on $\langle \kappa_{N,1} \rangle$ in 3d	146
	7.3.3 Bounds on $\langle \ \mathbf{u}\ _\infty \rangle$ and $\langle \ D\mathbf{u}\ _\infty^{1/2} \rangle$	148
7.4	The Kolmogorov length and intermittency	149
7.5	Singularities and the Euler equations	152

<i>Contents</i>		ix
7.6	References and further reading	155
	Exercises	155
8	Exponential decay of the Fourier power spectrum	157
8.1	Introduction	157
8.2	A differential inequality for $\ e^{\alpha t \nabla }\nabla\mathbf{u}\ _2^2$	157
8.3	A bound on $\ e^{\alpha t \nabla }\nabla\mathbf{u}\ _2^2$	163
8.4	Decay of the Fourier spectrum	165
8.5	References and further reading	167
	Exercises	167
9	The attractor dimension for the Navier-Stokes equations	169
9.1	Introduction	169
9.2	The $2d$ attractor dimension estimate	170
9.3	The $3d$ attractor dimension estimate	177
9.4	References and further reading	179
	Exercises	180
10	Energy dissipation rate estimates for boundary-driven flows	181
10.1	Introduction	181
10.2	Boundary-driven shear flow	182
10.3	Thermal convection in a horizontal plane	192
10.4	Discussion	197
10.5	References and further reading	203
	Exercises	204
Appendix A	Inequalities	205
	References	209
	Index	213