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Transferred to digital printing 2001

TAG

To

Paula

and

Sheila

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Preface

This book is not meant to be a review or a reference work, nor did we write it as a research monograph. It is not a text on fluid mechanics, and it is not an analysis course book. Rather, our goal is to outline one specific challenge that faces the next generation of applied mathematicians and mathematical physicists. The problem, which we believe is not widely appreciated in these communities, is that it is not at all certain whether one of the fundamental models of classical mechanics, of wide utility in engineering applications, is actually self-consistent.

The suspect model is embodied in the Navier-Stokes equations of incompressible fluid dynamics. These equations are nothing more than a continuum formulation of Newton's laws of motion for material "trying to get out of its own way." They are a set of nonlinear partial differential equations which are thought to describe fluid motions for gases and liquids, from laminar to turbulent flows, on scales ranging from below a millimeter to astronomical lengths. Only for the simplest examples are they exactly soluble, though, usually corresponding to laminar flows. In many important applications, including turbulence, they must be modified and matched, truncated and closed, or otherwise approximated analytically or numerically in order to extract any predictions. On its own this is not a fundamental barrier, for a good approximation can sometimes be of equal or greater utility than a complicated exact result.

The issue is that it has never been shown that the Navier-Stokes equations, in three spatial dimensions, possess smooth solutions starting from arbitrary initial conditions, even very smooth, physically reasonable initial conditions. It is possible that the equations produce solutions which exhibit finite-time singularities. If this occurs, then subsequent evolution may be nonunique, violating the fundamental tenets of Newtonian determinism for this model. Furthermore, finite-time singularities in the solutions signal that the equations are generating structures on arbitrarily small scales, contradicting the separation-of-scales assumption used to derive the hydrodynamic equations from microscopic models. It turns out that the nonlinear terms that can't be controlled mathematically are precisely those describing what is presumed to be the basic physical mechanism for the generation of turbulence, namely vortex stretching. So what may appear to applied scientists to be mathematical formalities, i.e., questions of existence and uniqueness and regularity, are actually intimately tied up with the efficacy of the Navier-Stokes equations as a model for fluid turbulence. Whether or not the equations actually do display these pathologies remains an open problem: It's never been proved one way or the other.

In this book we have tried to lay out the details of this quandary. In the first four chapters we introduce the Navier-Stokes equations together with some fundamental ideas about stability, turbulence, and dynamical systems in general. The remaining chapters deal with the associated mathematical issues, starting from what we can prove about existence and uniqueness and leading to the limits of what is known about their regularity. Our goals are to show how far we can go toward establishing regularity for solutions of the Navier-Stokes equations, to show some encouraging results of rigorous treatments of Navier-Stokes dynamics, and to show the shortcomings and limitations of the analysis. With the deepest respect and admiration for those who identified and initiated investigations into these mathematical issues, we recognize that this topic has remained closed to the mainstream of applied mathematics and mathematical physics, due in large part to the technical nature of the investigations, often phrased in the unfamiliar language of abstract functional analysis. The attempt we have made is to present some of the techniques and results of these studies in a familiar context, explaining and developing the tools as we proceed.

We have intended this book for graduate students and researchers in applied mathematics and theoretical/mathematical physics who want an introductory and detailed presentation of the methods, successes, and limitations of formal analysis of the Navier-Stokes equations. We presume a level of mathematical sophistication approximately at the level of a first year (UK) or second year (USA) mathematics graduate student, or of a similarly prepared physics or engineering student. The greatest part of the book is based on elementary ideas from Newtonian mechanics, real analysis, Fourier analysis, ordinary differential equations, and linear algebra. We have attempted to explain each step of the essential calculations and proofs, with the thought that this is what it takes not to lose or dishearten motivated, curious, and diligent readers.[1] In its entirety, this book would be a suitable text for a one semester graduate level special topics course. A selection of the material, say, Chapters 1-3, 5-6, and 8, comprising the essential elements of the statement and analysis of the problem, the methods and the results, could be covered in a normal British university term or American academic quarter.

Many people have contributed to our understanding of the contents of this book, and in particular we acknowledge fruitful discussions, collaborations, and/or interactions with M. Bartuccelli, C. Foias, D. Holm, S. Malham, and R. Temam. Marieka Fisher efficiently and skillfully assisted with the manuscript preparation. Special thanks go to Mark Alderson, James Robinson, and Edriss Titi who thoroughly and critically read through the manuscript. Finally, and most of all, we thank Peter Constantin and Dave Levermore who, over the last seven years, have patiently, expertly, and cheerfully taught us much of the material in this book, making the complicated simple, and the obscure clear. Much credit goes to them for what is worthwhile about this book. The blame for its flaws rests squarely on us.

[1] Many such readers will notice our preoccupation with the derivation and manipulation of inequalities, rather than equalities. At first this may seem to be taking the easy route to obtaining relatively weak results, but several comments are in order. First, it is often the case in theoretical physics and applied mathematics that "exact" calculations are performed for specific solutions of a problem, or for approximations of a problem. We are often concerned here with results for all (or a class of) solutions, without approximations. This means that bounds or "estimates," rather than exact values, are the only things that are realistically possible to prove. Furthermore, although one might think that it's easier to compute with (imprecise) inequalities than with (precise) equalities, actually it is just the opposite: With equalities there is one right answer, while with inequalities there are an infinite number of right answers. We are thus faced with the additional challenge of producing the best right answer!