1 The equations of motion

1.1 Introduction

The Navier-Stokes equations of fluid dynamics are a formulation of Newton's laws of motion for a continuous distribution of matter in the fluid state, characterized by an inability to support shear stresses. We will restrict our attention to the incompressible Navier-Stokes equations for a single component Newtonian fluid. Although they may be derived systematically from the microscopic description in terms of a Boltzmann equation, albeit with some additional fundamental assumptions, in this chapter we present a heuristic derivation designed to illustrate the elements of the physics contained in the equations.

1.2 Euler's equations for an incompressible fluid

First we consider an ideal inviscid fluid. The dependent variables in the so-called Eulerian description of fluid mechanics are the fluid density $\rho(\mathbf{x}, t)$, the velocity vector field $\mathbf{u}(\mathbf{x}, t)$, and the pressure field $p(\mathbf{x}, t)$. Here $\mathbf{x} \in \mathbf{R}^d$ is the spatial coordinate in a d-dimensional region of space (d typically takes values 2 or 3, with a default value of 3 in this chapter). An infinitesimal element of the fluid of volume δV located at position $\mathbf{x}$ at time t has mass $\delta m = \rho(\mathbf{x}, t)\delta V$ and is moving with velocity $\mathbf{u}(\mathbf{x}, t)$ and momentum $\delta m\, \mathbf{u}(\mathbf{x}, t)$. The normal force directed into the infinitesimal volume across a face of area $\mathbf{n}\,\delta a$ centered at $\mathbf{x}$, where $\mathbf{n}$ is the outward directed unit vector normal to the face, is $-\mathbf{n}p(\mathbf{x}, t)\delta a$. The pressure is the magnitude of the force per unit area, or normal stress, imposed on elements of the fluid from neighboring elements. These definitions are illustrated in Figure 1.1.

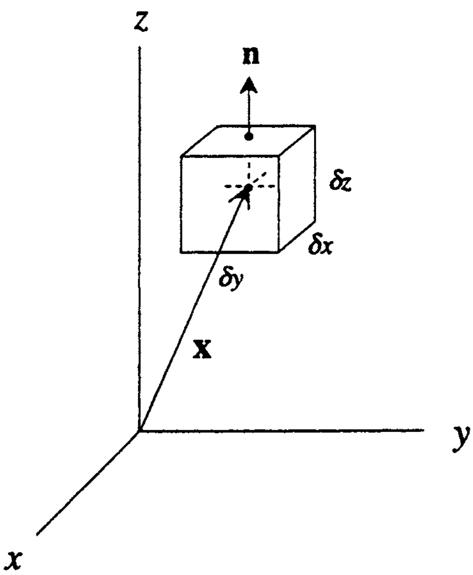

Fig. 1.1. A fluid element of volume $\delta V = \delta x \delta y \delta z$ located at position $\mathbf{x}$. The top surface's outward pointing normal $\hat{\mathbf{n}}$ is shown.

The fundamental kinematic principle is contained in the notion of the convective derivative. On the one hand, the rate of change of a quantity given by the function $f(\mathbf{x}, t)$ at a fixed point $\mathbf{x}$ in space is simply the partial derivative with respect to time:

$$\left(\frac{df(\mathbf{x},t)}{dt}\right)_{\text{fixed position}} = \lim_{\delta t \to 0} \frac{f(\mathbf{x}, t+\delta t) - f(\mathbf{x}, t)}{\delta t} = \frac{\partial f(\mathbf{x},t)}{\partial t}. \tag{1.2.1}$$

On the other hand, the rate of change of the same quantity at $\mathbf{x}$, as measured by an observer moving with velocity $\mathbf{u}$, is

$$\begin{aligned}\left(\frac{df(\mathbf{x},t)}{dt}\right)_{\text{moving}} &= \lim_{\delta t \to 0} \frac{f(\mathbf{x} + \mathbf{u}\delta t, t+\delta t) - f(\mathbf{x}, t)}{\delta t} \\ &= \frac{\partial f(\mathbf{x},t)}{\partial t} + \mathbf{u} \cdot \nabla f(\mathbf{x},t). \end{aligned} \tag{1.2.2}$$

We refer to this rate of change with respect to an observer moving with the fluid, as the *convective derivative* and denote it by d/dt. That is, for a function of both $\mathbf{x}$ and t,

$$\frac{df(\mathbf{x},t)}{dt} := \frac{\partial f(\mathbf{x},t)}{\partial t} + \mathbf{u} \cdot \nabla f(\mathbf{x},t). \tag{1.2.3}$$

There is no ambiguity in the definition of the time derivative for functions of time alone, where the standard notation d/dt will be used.

The fundamental equations of motion for a fluid system characterized by $\rho(\mathbf{x},t)$, $\mathbf{u}(\mathbf{x},t) = (u_1, u_2, u_3)$ and $p(\mathbf{x},t)$ come from three different distinct considerations: conservation of mass, Newton's second law, and material properties.

Consider the volume δV of an element of mass δm as the system evolves. Conservation of mass means that δm doesn't change for this element. If the element compresses or expands then the volume and density will change, but the mass is fixed:

$$\frac{d\delta m}{dt} = 0. \tag{1.2.4}$$

The rate of change of the volume occupied by δm is obtained as follows. For a rectangular volume $\delta V = \delta x \delta y \delta z$ we write

$$\frac{d\delta V}{dt} = \frac{d\delta x}{dt}\delta y \delta z + \frac{d\delta y}{dt}\delta x \delta z + \frac{d\delta z}{dt}\delta x \delta y. \tag{1.2.5}$$

The length elements increase or decrease according to the relative velocity of their endpoints. The rate of change of the length δx is

$$\frac{d\delta x}{dt} = u_1(x + \delta x/2, y, z, t) - u_1(x - \delta x/2, y, z, t) = \frac{\partial u_1}{\partial x}\delta x, \tag{1.2.6}$$

and likewise for the other components. Combined with equation (1.2.5), this gives

$$\frac{d\delta V}{dt} = (\nabla \cdot \mathbf{u})\,\delta V. \tag{1.2.7}$$

Hence the divergence of the velocity vector field is the local rate of change of the volume of elements of mass. In terms of the density ρ,

$$\frac{d\rho}{dt} = \frac{d}{dt}\frac{\delta m}{\delta V} = -\frac{\delta m}{(\delta V)^2}\frac{d\delta V}{dt} = -\rho \nabla \cdot \mathbf{u}. \tag{1.2.8}$$

Using the definition of the convective derivative, we see that conservation of mass manifests itself as the *continuity equation*

$$0 = \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho \nabla \cdot \mathbf{u} = \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}). \tag{1.2.9}$$

Newton's second law of motion, which states that the rate of change of momentum equals the net applied force, can be applied to each element

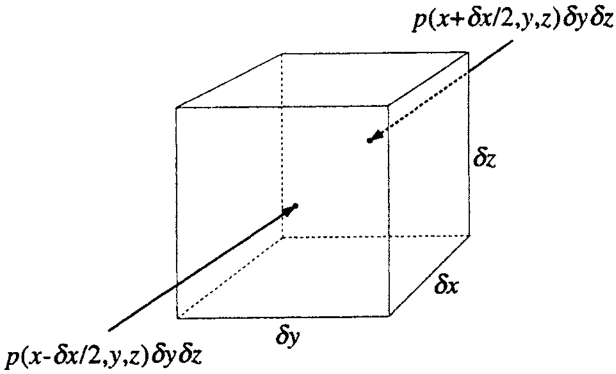

Fig. 1.2. The pressure force acting on the front and rear faces of a fluid element.

of mass in the fluid. In the absence of any externally applied forces, the net force δF acting on each element of mass is due to the pressure field. The component of force in the x-direction, as illustrated in Figure 1.2, is

$$\delta F_1 = p\left(\mathbf{x} - \hat{\mathbf{i}}\delta x/2, t\right)\delta y \delta z - p\left(\mathbf{x} + \hat{\mathbf{i}}\delta x/2, t\right)\delta y \delta z = -\frac{\partial p}{\partial x}\delta V. \quad (1.2.10)$$

Similar expressions hold for the y and z components of the force. Hence Newton's second law for the element of fluid mass δm at position δx is,

$$\frac{d}{dt}(\delta m\, \mathbf{u}(\mathbf{x}, t)) = \delta \mathbf{F} = -\delta V\, \nabla p. \quad (1.2.11)$$

Recalling the equation of conservation of mass (1.2.4) and the definition of the convective derivative (1.2.3) and dividing through by δm we find *Euler's equations*

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho}\nabla p. \quad (1.2.12)$$

Combined, the continuity equation (1.2.9) and Euler's equations (1.2.12) provide $d+1$ evolution equations for the $d+2$ dependent variables (ρ, p and the d components of $\mathbf{u}$). What remains is to provide a connection between the density and pressure. Typically this is in the form of a thermodynamic equation of state. For example, in an ideal gas at constant temperature, $p \sim \rho$. If temperature variations are to be accounted for, then the pressure may become a function of both the local density and the local temperature and a further evolution equation for the temperature must be supplied. This matter will be taken up in section 1.5. A significant simplification is achieved by considering fluids which are effectively incompressible. Mathematically the condition of incompressibility is simply

$$\nabla \cdot \mathbf{u} = 0. \tag{1.2.13}$$

Physically, this constraint restricts applicability to problems where all the relevant velocities are much less than the speed of sound in the fluid. The continuity equation (1.2.9) then implies that the convective derivative of the density vanishes, so the density of each fluid element never changes from its initial value. This, in turn, implies that an initially homogeneous (constant density) fluid remains so:

$$\rho(\mathbf{x}, 0) = \text{constant} \qquad \Rightarrow \qquad \rho(\mathbf{x}, t) = \text{constant}. \tag{1.2.14}$$

Euler's equations for an incompressible homogeneous fluid are

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{1}{\rho} \nabla p = 0 \tag{1.2.15}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.2.16}$$

where the density is now a parameter. These are $d + 1$ equations for the $d+1$ unknowns (p and the d components of $\mathbf{u}$). The pressure is determined by the velocity vector field, as is seen by taking the divergence of (1.2.15), commuting the space and time derivatives, and using the divergence-free condition on $\mathbf{u}$ from (1.2.16):

$$\Delta p = -\rho \nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) = -\rho\, u_{i,j} u_{j,i}. \tag{1.2.17}$$

The pressure field is a solution of Poisson's equation with a source which is quadratic in the derivatives of the velocities; the pressure is a nonlocal functional of the instantaneous flow configuration. As a result of the incompressibility condition, the pressure is the stress applied by neighboring parts of the fluid on one another in an attempt to "push each other out of the way." These forces are propagated instantaneously (the speed of sound is effectively infinite) and correlated over long ranges (the kernel of the inverse of the Laplacian decays as r^{-1} in three dimensions). Combining equations (1.2.15) and (1.2.17), Euler's equations may be thought of as a nonlocal evolution equation for **u** alone.[1] They are not completely posed, however, until appropriate boundary conditions are specified. Boundary conditions are necessary for both **u** in (1.2.15) and p in (1.2.17), and are determined by the physics of the problem at hand.

If the fluid is confined to a fixed region of space Ω bounded by a stationary boundary $\partial\Omega$, then the fluid cannot cross these rigid boundaries. This means that the normal component of the velocity vector field satisfies

$$\mathbf{n}\cdot\mathbf{u}|_{\partial\Omega} = 0, \tag{1.2.18}$$

where **n** is the local normal to $\partial\Omega$. To derive boundary conditions for Poisson's equation for the pressure we consider the normal component of (1.2.15) on the boundary. Because, formally,

$$\mathbf{n}\cdot\left.\frac{\partial\mathbf{u}}{\partial t}\right|_{\partial\Omega} = 0 \tag{1.2.19}$$

the pressure satisfies the Neumann boundary conditions

$$\mathbf{n}\cdot\nabla p\,|_{\partial\Omega} = -\rho\,\mathbf{n}\cdot[\mathbf{u}\cdot\nabla\mathbf{u}]|_{\partial\Omega}\,. \tag{1.2.20}$$

For stationary flat boundary surfaces, e.g., the walls of a fixed rectangular box, this condition simplifies to

$$\mathbf{n}\cdot\nabla p|_{\partial\Omega} = 0. \qquad \text{(Euler)} \tag{1.2.21}$$

In these cases these boundary conditions are sufficient to determine p uniquely up to an additive constant (and explicitly, if the relevant Green's functions are known explicitly) for a given instantaneous velocity field **u**.

Another set of boundary conditions, often used in both theoretical and numerical studies, are periodic boundary conditions where the fluid motion is confined to a 2-torus or 3-torus. Mathematically one imposes periodic boundary conditions on p and each component of **u**, in each of the d directions, with spatial periods $L_1, ..., L_d$. The technical advantage of periodic boundary conditions is that they allow for the study of a

[1] This is the view taken in the mathematical literature. Together, (1.2.15), (1.2.16), and (1.2.17) may be written in the compact form

$$\mathbf{P}\left\{\frac{d\mathbf{u}}{dt}\right\} = 0,$$

where the operator **P** is a projection onto divergence-free vector fields. For example, on a torus in d dimensions, the expression for the projection is

$$\mathbf{P}\{\mathbf{v}\} = \mathbf{v} - \nabla\Delta^{-1}(\nabla\cdot\mathbf{v}),$$

where Δ^{-1} is the inverse Laplacian with the appropriate periodic boundary conditions. In this notation the pressure is explicitly recognized as an artifact of the incompressibility condition.

finite volume of fluid both with translation invariance and without either the physical or mathematical complications of rigid boundaries. The implications of these complications will become apparent in later chapters. For either choice of boundary conditions, the mathematical challenge is to show that along with appropriate initial conditions (say, the flow configuration $\mathbf{u}(\mathbf{x}, 0)$ at $t = 0$), Euler's equations are an apparently well posed problem for the time evolution of the velocity vector field.

1.3 Energy, body forces, vorticity, and enstrophy

Consider an ideal incompressible fluid in the volume Ω bounded by $\partial\Omega$, with either rigid or periodic boundary conditions. The kinetic energy in the fluid is

$$\int_\Omega \left(\frac{1}{2}\rho\,|\mathbf{u}|^2\right) d^dx = \frac{1}{2}\rho\,\|\mathbf{u}\|_2^2, \tag{1.3.1}$$

where we have introduced the notation $\|\cdot\|_2$ for the norm in $L^2(\Omega)$, the Hilbert space of square integrable functions. The usual law of conservation of energy applies to an ideal incompressible fluid, and it can be derived from Euler's equations. Differentiating $\frac{1}{2}\rho\,\|\mathbf{u}\|_2^2$ with respect to time and using the equation of motion (1.2.15), we find

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}\rho\,\|\mathbf{u}\|_2^2\right) &= \rho\int_\Omega \mathbf{u}\cdot\frac{\partial\mathbf{u}}{\partial t}\,d^dx \\ &= -\rho\int_\Omega \mathbf{u}\cdot\left[\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p\right] d^dx. \end{aligned} \tag{1.3.2}$$

Noting the divergence-free condition on $\mathbf{u}$,

$$\mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) = \frac{1}{2}\mathbf{u}\cdot\nabla|\mathbf{u}|^2 = \frac{1}{2}\nabla\cdot\left(\mathbf{u}|\mathbf{u}|^2\right) \tag{1.3.3}$$

and

$$\mathbf{u}\cdot\nabla p = \nabla\cdot(\mathbf{u}p). \tag{1.3.4}$$

The divergence theorem applied to the last line of (1.3.2) yields

$$\frac{d}{dt}\left(\frac{1}{2}\rho\,\|\mathbf{u}\|_2^2\right) = -\int_{\partial\Omega}\left(\frac{1}{2}\rho\,|\mathbf{u}|^2 + p\right)\mathbf{u}\cdot\mathbf{n}\,da. \tag{1.3.5}$$

For stationary rigid boundary conditions $\mathbf{u}\cdot\mathbf{n}$ vanishes on $\partial\Omega$ so the surface integral vanishes. And for periodic boundary conditions, any such "surface" integral vanishes identically (because there really is no surface). Hence in either case,

$$\frac{d}{dt}\left(\frac{1}{2}\rho\,\|\mathbf{u}\|_2^2\right) = 0 \tag{1.3.6}$$

and the total kinetic energy is conserved. This is natural because we have not taken any dissipative effects into account, and there is no external work being done on the system. Mathematically, we say that the L^2 norm is conserved.

If an external force is applied to the fluid, then Euler's equations for an incompressible fluid become

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = \frac{1}{\rho}\,\mathbf{f}(\mathbf{x},t) \tag{1.3.7}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.3.8}$$

where the "body force" $\mathbf{f}(\mathbf{x},t)$ is the applied force per unit volume. The kinetic energy then evolves according to

$$\frac{d}{dt}\left(\frac{1}{2}\rho\,\|\mathbf{u}\|_2^2\right) = \int_\Omega \mathbf{u}\cdot\mathbf{f}\,d^dx. \tag{1.3.9}$$

The source or sink of energy in (1.3.9) is the power expended or absorbed by the body force.

If the force field is the gradient of a potential per unit volume, i.e., if

$$\mathbf{f} = -\nabla\phi, \tag{1.3.10}$$

then the net work done by the body force vanishes identically for either stationary rigid or periodic boundary conditions:

$$\int_\Omega \mathbf{u}\cdot(-\nabla\phi)\,d^dx = -\int_{\partial\Omega}\phi\mathbf{u}\cdot\mathbf{n}\,da. \tag{1.3.11}$$

In the case of a gradient body force, the body force can be absorbed into the pressure term. The evolution equation (1.3.7) is simply

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla(p+\phi) = 0, \tag{1.3.12}$$

and the solution for $\mathbf{u}(\mathbf{x},t)$ is the same whether or not ϕ is present; the potential just renormalizes the pressure field. Hence when we include external body forces, we will typically consider nongradient fields, i.e., those with a nonvanishing curl in a simply connected domain.

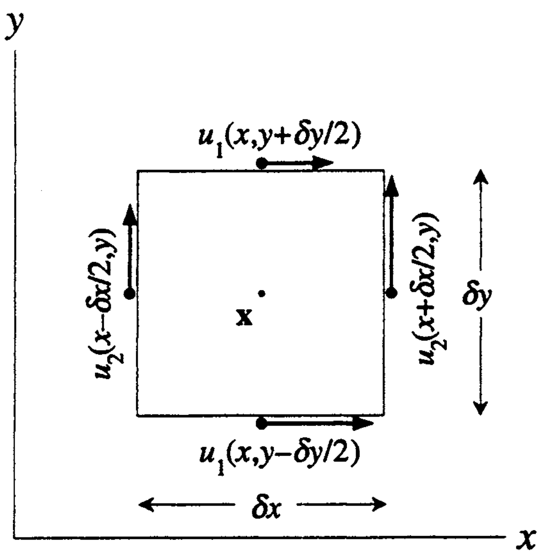

Fig. 1.3. Components of the velocity on the faces of a fluid element contributing to the average angular velocity in the z-direction.

Besides the local velocity of the fluid, another important kinematic quantity is the local angular velocity. The standard measure of this angular velocity is the vorticity $\boldsymbol{\omega}$, defined by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u}. \tag{1.3.13}$$

To see how this is related to the angular velocity of a piece of the fluid, consider a small rectangular mass δm located at $\mathbf{x}$ as in Figure 1.3. Then the average z-component, $\langle \Omega_3 \rangle$, of the angular velocity of the particle's edges about the point $\mathbf{x}$ is

$$\begin{aligned} \langle \Omega_3 \rangle &= \frac{1}{4}\left[\frac{u_2\left(x+\delta x/2, y\right)}{\delta x/2} - \frac{u_1\left(x, y+\delta y/2\right)}{\delta y/2} \right. \\ &\quad \left. - \frac{u_2(x-\delta x/2, y)}{\delta x/2} + \frac{u_1(x, y-\delta y/2)}{\delta y/2}\right] \\ &= \frac{1}{2}\left[\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right] \\ &= \frac{1}{2}\omega_3. \end{aligned} \tag{1.3.14}$$

An evolution equation for the vorticity is derived by taking the curl of

Euler's equation for the velocity vector field. Using standard $3d$ vector calculus identities along with the divergence free condition on $\mathbf{u}$ we obtain, from (1.2.15) and (1.2.16),

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}. \tag{1.3.15}$$

(If a body force $\mathbf{f}$ is present, then there is also a $\rho^{-1}\nabla \times \mathbf{f}$ on the right-hand side above.) The vorticity evolution equation contains some very important physics which plays a fundamental role in understanding the challenges of fluid turbulence and its mathematical manifestations. The complex fluid motions associated with turbulent flows are often described in terms of "eddies," which are the vortices – local concentrations of vorticity – in a fluid. The vorticity equation also provides the first indication of the fundamental difference between fluid flows in two and three spatial dimensions.

First consider the $2d$ problem. If the velocity vector field is confined to the $x - y$ plane, where $\mathbf{u}(\mathbf{x}, t) = \hat{\mathbf{i}}u_1(x, y, t) + \hat{\mathbf{j}}u_2(x, y, t)$, then the curl only has a z component ω_3, which we will unambiguously refer to as the scalar ω. Only the z component of (1.3.15) does not vanish identically, although every component of the $\boldsymbol{\omega} \cdot \nabla \mathbf{u}$ term does vanish identically. In $2d$ the scalar vorticity ω evolves according to

$$\frac{\partial \omega}{\partial t} + \mathbf{u} \cdot \nabla \omega = 0, \tag{1.3.16}$$

which is more transparently written in terms of the convective derivative,

$$\frac{d\omega}{dt} = 0. \tag{1.3.17}$$

The vorticity of each particle of the fluid is a constant of the motion. There is no internal mechanism in the $2d$ Euler's equations for angular velocity to be transferred between different parts of the fluid. Individual parts of the fluid are transported by the flow field $\mathbf{u}$, but the local angular velocity associated with each part remains the same. Initially distinct vortices (localized concentrations of vorticity) are simply transported in the resulting flow field, interacting in possibly complicated ways, but maintaining their identity both in form and in magnitude.

The full $3d$ problem is very different. In terms of the convective derivative, the vorticity equation (1.3.15) is

$$\frac{d\boldsymbol{\omega}}{dt} = \boldsymbol{\omega} \cdot \nabla \mathbf{u}. \tag{1.3.18}$$

The $\boldsymbol{\omega}\cdot\nabla\mathbf{u}$ term gives rise to a phenomenon referred to as *vortex stretching*.