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127 Spectral Theory of the Riemann Zeta-Function

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Contents

Preface

EVER since Riemann's use of the theta transformation formula in one of his proofs of the functional equation for the zeta-function, number-theorists have been fascinated by various interactions between the zeta-function and automorphic forms. These experiences, however, have remained episodic like rare glimpses of crests, for most of them ensued from apparently spontaneous relations of the zeta-function with a variety of Eisenstein series. Nevertheless such glimpses are highly suggestive of a grand view over and far beyond the Eisenstein ridge, and bring forth the notion of a *kamuy-mintar* where the entire collection of automorphic forms contribute to the formation of the zeta-function.

My aim in the present monograph is to try to substantiate this belief by demonstrating that the zeta-function has indeed a structure tightly supported by all automorphic forms. The story begins with an unabridged treatment of the spectral resolution of the non-Euclidean Laplacian, and continues to a theory of trace formulas. The fundamental means thus readied are subsequently mustered up for the quest to find an explicit formula for the fourth power moment of the zeta-values. Then the zeta-function emerges as a magnificent peak embracing infinitely many gems called automorphic L-functions representing the spectrum.

My best thanks are due to my friends A. Ivić and M. Jutila for their unfailing encouragement, and to D. Tranah, P. Jackson, and all of the personnel of the Cambridge University Press engaged in this project for sharing their professional vigor. I must also thank my family for the comfort and the music that have been sustaining my scientific life.

Tokyo
January, 1997 — Y. M.

Convention and assumed background

Once introduced most symbols will remain effective throughout the sequel. Some of them are naturally standard. Thus $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ are sets of all integers, rationals, reals, and complex numbers, respectively. For example the group composed of all $n \times n$ integral matrices with determinant equal to 1 is denoted by $SL(n, \mathbb{Z})$. The arithmetic functions $\sigma_a(n)$ and $d_k(n)$ stand, respectively, for the sum of the ath powers of divisors of n and for the number of ways of expressing n as a product of k integral factors. In particular, $d(n) = d_2(n)$ is the divisor function. The Bessel functions are denoted by I_ν, J_ν, K_ν as usual. We use the term K-Bessel function to indicate K_ν without the specification of the order ν; and the same convention applies to other Bessel functions as well. The symbol Γ is for the gamma function, and $\mathit{\Gamma}$ is for the full modular group introduced in Section 1.1. The dependency of implied constants on others will not always be explained, since it is more or less clear from the context.

Some knowledge of integrals involving basic transcendental functions is certainly helpful. For this purpose Lebedev's book [38] is quite handy. But there are occasions when Titchmarsh [69], Watson [74], and Whittaker and Watson [75] give more precise information, though proofs of most integral formulas and relevant estimates are given or at least briefly indicated either in the text or in the respective notes. In addition to these standard books, Sonine's article [68], Vilenkin's book [71], and the table [15] by Gradshteyn and Ryzhik are recommended.

Readers are supposed to have ample knowledge of the zeta-function such as that developed in Titchmarsh [70] as well as in Ivić [19]. In fact this monograph is, in part, a continuation of their books. Thus, for instance, bounds like

$$(\log t)^{-1} \ll |\zeta(1+it)| \ll \log t, \qquad \zeta(\tfrac{1}{2}+it) \ll t^{\frac{1}{6}} \log t,$$

$$\int_0^T |\zeta(\tfrac{1}{2}+iu)|^4 du \ll T(\log T)^4 \quad (t,\ T \geq 2)$$

are used freely under the term *classical estimates*. On the other hand no experience in the theory of automorphic functions is assumed. The first three chapters can be taken for an introduction to the subject.

The references are limited to the essentials. Suggestions for further readings may be found in the notes and the articles quoted there.