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Introduction

Let X and Y be real separable Banach spaces, D a subset of X, $T: D \subseteq X \to Y$ a possibly nonlinear mapping, and $\Gamma_0 = \{X_n, P_n; Y_n, Q_n\}$ a suitable approximation scheme for the equation

$$\text{(I)} \qquad T(x) = y, \quad x \in D, \ y \in Y.$$

One of the basic problems of functional analysis is to solve Eq. (I) and to study the nature of the solution set of (I). In this book we distinguish in principle between two notions of solvability of Eq. (I), namely:

(A) *Solvability* of Eq. (I), in which existence of a solution is somehow established; and

(B) *Approximation-solvability* (A-solvability, for short) of Eq. (I), in which a solution x of (I) is obtained as a limit (or at least as a limit point) of solutions x_n of simpler finite dimensional equations

$$\text{(II)} \qquad T_n(x_n) = Q_n y, \quad x_n \in D_n \equiv D \cap X_n, \ Q_n y \in Y_n, \ T_n \equiv Q_n T|_{D_n}.$$

If x and x_n are unique solutions, then (I) is said to be *uniquely A-solvable*.

In the classical functional analysis, problem (I) has been handled satisfactorily if a given equation is reducible to one in which T is either of the form $T = I - S$ with $S: D \subseteq X \to X$ contractive, or $T = I - C$ with $C: D \subseteq X \to Y$ compact. The contraction mapping principle, Schauder fixed point theorem, Leray–Schauder degree theory for $T = I - C$, Galerkin method for $T = I - C$, and their consequences provided the basis for the treatment of Eq. (I) for these special classes of mappings. For extensive literature of this classical case see Leray and Schauder [57], Cronin [S], Deimling [24c], Krasnoselsky [51b], Rothe [82a,c], and Zeidler [112b].

The main thrust of the recent development of nonlinear functional analysis has been in two directions. The first involves breaking out of the classical framework into a much wider class of noncompact operators: monotone by Minty and Browder; pseudomonotone by Brezis and Lions; nonexpansive by Kirk and Browder; ball- and set-condensing by Sadovsky and Nussbaum; P-compact and A-proper by Petryshyn; types (S) and (S_+) by Browder; (α)-maps by Skrypnik; quasilinear Fredholm maps by Fitzpatrick and Pejsachowicz; and others. The second direction is the development of *coincidence degree* theory, by Mawhin for the case of L-compact maps and by Hetzer for L-condensing maps, and the *alternative method* by Cesari, Hale, Kannan, McKenna, and others as well as hyperbolic A-proper mappings by Pascali and Milojevič. For descriptions of most of these classes of maps, see the monographs [12; 14h; 17; 24c; 33c; 38; 47; 52; 58; 59; 68b; 73r; 93d; 102; 112a,b]. See also [49; 66e; 70a,c; 93b; 108b], Ambrosetti and Prodi [S], and Brezis [S].

The notion of an A-proper mapping introduced by the writer in 1967 (see [73r] for details) came about as an answer to the following problem: For what type of linear or nonlinear mapping T is it possible to construct a solution of Eq. (I) as a strong limit of solutions x_n of Eq. (II)? This problem was studied in a series of papers, and the notion which evolved from these investigations is that of an *A-proper mapping* (see [73a,d]).

The A-properness of T is not only closely connected with A-solvability of Eq. (I) but is also coextensive. Thus, for example, it was shown by the author [73r] that if $T \in L(X, Y)$, then Eq. (I) is uniquely A-solvable with respect to Γ_0 if and only if T is A-proper with respect to Γ_0 and one-to-one. Further, if either X or Y is reflexive, then Eq. (I) is uniquely A-solvable with respect to Γ_0 if and only if there exists a constant $\gamma > 0$ and $N_0 \in \mathbb{Z}^+$ such that $\|Q_n Tx\| \geq \gamma\|x\|$ for all $x \in X_n$ and each $n \geq N_0$. These characterization theorems are best possible; they contain not only earlier results for Galerkin and Petrov–Galerkin methods obtained in the Soviet Union, but also provided a powerful stimulus for the development of the theory of nonlinear A-proper mappings. Because the class of these mappings is quite broad, the theory of A-proper maps both extends earlier results concerning Galerkin-type methods for nonlinear equations and unifies them with more recent results in the theory of strongly monotone and accretive operators, operators of type (S) and (S_+), P_γ-compact, ball-condensing, L-compact, and other mappings. The class of A-proper maps has thus become a subject of extensive study. In addition to the author, the numerous and important contributions by Fitzpatrick and Webb are particularly noteworthy. Many single-valued results were extended by Milojevič to multi-valued maps. The topological degree theory for A-proper operators was developed by Browder and Petryshyn [16a,b]. The references to the contributions of many other authors will be given in the text. Successful

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applications of the A-proper mapping theory to some open problems in physics were carried out by Chandler and Gibson [19], Kröger and Perne [53], and others. In [82b] Rothe established the existence of the relationship between A-proper maps and the alternative method by proving that the Cesari index for $T = L - N$, where N is L-compact in the sense of Mawhin [63a], is equal to the Browder–Petryshyn degree under judicious choice of the approximation scheme. His result was simplified and improved by Willem [108a]. For recent applications to ODEs and PDEs, see [33c; 49; 52; 54; 66d; 70b; 73s; 76a–g; 93d; 104b; 105a,b; 107a].

Skrypnik's 1973 monograph [93a] developed a new topological degree theory for maps of type (α) from a reflexive space X to its dual X^*, and applied it to the solvability of nonlinear elliptic PDEs of abstract and concrete nature. Since the bounded map of type (α) is the same as the mapping of type (S_+) of Browder [14f] and since both are A-proper, as was shown independently in [73b] and [14h], Skrypnik's theory is related to the A-proper mapping theory.

The basic theory of A-proper maps developed before the early 1980s was outlined by the author in [73r]. One of the main purposes of this monograph is to use the topological degree for densely defined A-proper operators in the systematic study of the solvability and/or approximation-solvability of the semilinear equation

$$\text{(III)} \qquad Lx - Nx = y, \quad x \in \bar{G}_D, \ y \in Y,$$

where $L: D(L) \subset X \to Y$ is a Fredholm mapping of index $i(L) \geq 0$, $G \subset X$ is open, and $N: \bar{G}_D \equiv \bar{G} \cap D \subset X \to Y$ is a nonlinear mapping such that $L - N$ or $T_\lambda \equiv L - \lambda N: \bar{G}_D \subset\subset X \to Y$ is A-proper for each $\lambda \in (0,1]$ with respect to a suitable approximation scheme. Using a variant of a basic result in [32, Thm. 1.2], we shall present general results concerning the structure of the solution set of (III) which contain the earlier ones of Amann, Ambrosetti, and Mancini [5], Furi and Pera [37a], Massabo and Pejsachowicz [62], Mawhin and Rybakowski [S], Pera [71], Petryshyn [73s], Rabinowitz [79b], Zhang [113], and others.

Using the notion of parity introduced and studied by Fitzpatrick and Pejsachowicz [33b] and adopting the arguments of [31d], we shall also develop the global bifurcation theory essentially due to Fitzpatrick [31d] for the equation

$$\text{(IV)} \qquad L(\lambda)(x) - N(\lambda, x) = 0, \quad (\lambda, x) \in J \times \bar{D},$$

with $D \subset X$ open and $J \subset \mathbb{R}$, and where $L(\lambda): J \to \Phi(X, Y)$ is continuous and A-proper for each $\lambda \in J$. The results presented here contain the earlier ones of Krasnoselsky [51a], Pascali [70b], Petryshyn [73g], Rabinowitz

[79a], Toland [100a,b], Webb and Welsh [105b], Welsh [107a,b], and others. See Ize [45] for an exhaustive survey.

Our abstract results (many of them new ones) obtained for Eq. (III) will be applied to the solvability and/or A-solvability of semilinear ordinary and partial elliptic differential equations for which the classical theory based on degrees for compact or condensing fields is not applicable. The study of quasilinear PDEs via the A-proper mapping theory was initiated by the author in 1975 [73e]. The latter theory was then also used in [73h] to extend some results for semilinear ODEs and PDEs obtained in [31b] via the condensing mapping theory. Subsequently, the A-proper mapping theory was used in [32; 34b,c; 66b,d; 73l,n,r,s; 76a–g; 104g; 105a; 107a] and elsewhere to obtain new results for semilinear equations.

As we shall see, our results will include as special cases the classical results when, in Eq. (III), the partial inverse L_1^{-1} of L or N is compact. The latter class of equations has been extensively studied by many authors using the Leray–Schauder degree or Schauder fixed point theorem (see [55a; 68a; 71; 90]), the alternative method as developed by Cesari and others in [17; 35; 38; 41; 63a,c], and the coincidence degree of Mawhin [63a]. However, the existence results for Eq. (III) when neither L_1^{-1} nor N is compact were first obtained under different conditions in the late 1970s by Hetzer [44b] and Fitzpatrick [31b] when N is condensing, by Brezis and Nirenberg [12] when N is of monotone type, and by Petryshyn [73h] when N is such that $L-N: X \to Y$ is A-proper or even weakly A-proper or pseudo-A-proper.

Beginning with Leray and Schauder [57], the topological degree approach to studying problems (III) and (IV) under various conditions on N and L was used by a great number of authors. Although works of many writers will be discussed in this monograph, for a more complete list of contributors the reader should consult the books of Berger [S], Deimling [24c], Fitzpatrick and Pejsachowicz [33c], Gaines and Mawhin [38], Joshi and Bose [47], Nirenberg [68b], Petryshyn [73r], Skrypnik [93d], and Zeidler [112b]. Among others, those conditions sufficient to ensure the existence of solutions to (III) when N satisfies both an asymptotic growth condition and an asymptotic positivity condition will be studied. Such conditions have their origin in the existence results for elliptic PDEs of Landesman and Lazer [55a], and have since been considered by many authors including Amann, De Figueiredo, Fitzpatrick, Fučik, Furi, Gossez, Gupta, Hess, Hetzer, Iarušek, Martelli, Massabo, Mawhin, Milojevič, Nirenberg, Pejsachowicz, Pera, Petryshyn, Schechter, Shaw, Vignoli, Webb, Williams, and Zeidler.

1

Introduction to the Brouwer and Leray–Schauder Degrees, A-Proper Mappings, and Linear Theory

In the first part of this chapter we outline the Brouwer degree theory, since it is needed to define and develop the theory of the generalized topological degree of A-proper mappings. We then state the definition and indicate some needed properties of the Leray–Schauder degree. In the second part we introduce the notion of an A-proper mapping, give some examples, and establish those properties which will be needed in subsequent chapters.

1.1. Definition of the Brouwer Degree for C Functions in $\mathbb{R}^N$ and Some Properties

This section is devoted to defining the Brouwer degree of a continuous function φ defined on the closure $\bar{D}$ of an open bounded subset D of $\mathbb{R}^N$. However, we first state the definition for functions in $C^1(\bar{D})$ which approximate the given C function. Our outline follows that of Schwartz [S] and Lloyd [59].

Let $\varphi \in C^1(\bar{D})$. We call $x \in D$ a *critical point* of φ if $J_\varphi(x) = 0$, where $J_\varphi(x)$ is the Jacobian determinant of φ at x (i.e., $J_\varphi(x) = \det(\varphi'(x)) = \det(\partial\varphi_i/\partial x_j)$); then $\varphi(x)$ is called a *critical value* of φ. The set of critical values of φ in $\bar{D}$ is denoted by $Z_\varphi(\bar{D})$, or simply by Z_φ; the set of critical values Z_φ is called the *crease* of φ.

Because $\bar{D}$ is compact, it is easy to show that if $\varphi \in C^1(\bar{D})$ and $p \notin \varphi(Z_\varphi)$, then $\varphi^{-1}(p)$ is finite. We can define the degree of φ at p when φ is a C^1 function and p is not a critical value of φ: It is the algebraic number of points x in D for which $\varphi(x) = p$; it counts $+1$ or -1 at p point x according as φ is orientation-preserving or orientation-reversing near x.

DEFINITION 1.1. Let $\varphi \in C^1(\bar{D})$, $p \notin \varphi(\partial D)$, and $p \notin \varphi(Z_\varphi)$. Define the *topological degree* of φ at p relative to D to be $\deg(\varphi, D, p)$, where

(1.1) $$\deg(\varphi, D, p) = \sum_{x\in\varphi^{-1}(p)} \operatorname{sign} J_\varphi(x).$$

Note that the summation in (1.1) is finite since, as just observed, $\varphi^{-1}(p)$ is finite when $p \notin (Z_\varphi)$. Usually, in this case the point $x \in \varphi^{-1}(p)$ is called a *regular point* of φ in D, while $p = \varphi(x)$ is called the *regular value* of φ. It should be added that the condition $p \notin \varphi(\partial D)$ is essential; it cannot be removed.

The following is clear from Definition 1.1: If I is the identity mapping, then

$$\deg(I, D, p) = \begin{cases} 1 & \text{if } p \in D, \\ 0 & \text{if } p \notin D. \end{cases}$$

In order to remove the condition that $p \notin \varphi(Z_\varphi)$ we use the method of approximation; for that, the following is essential.

PROPOSITION 1.1. *Let $\varphi \in C^1(\bar{D})$ and $p \notin \varphi(\partial D) \cup \varphi(Z_\varphi)$. Then there is a $\delta > 0$, depending on p and φ, such that if $\|\psi - \varphi\|_1 < \delta$ then $p \notin \psi(Z_\psi) \cup \psi(\partial D)$ and $\deg(\psi, D, p) = \deg(\varphi, D, p)$, where $\|\cdot\|_1$ denotes the norm in $C^1(\bar{D})$.*

As was shown by Schwartz [S], using the approach of Heinz [42] (where the $\deg(\varphi, D, p)$ is defined by a suitable integral) and the theorem of Sard [87], one can remove the restriction in Definition 1.1 that $p \notin \varphi(Z_\varphi)$ and so state the general definition of degree for a C^1 function.

DEFINITION 1.2. If $\varphi \in C^1(\bar{D})$ and $p \notin \varphi(\partial D)$ but $p \in \varphi(Z_\varphi)$, then define $\deg(\varphi, D, p)$ to be $\deg(\varphi, D, q)$, where q is any point such that $q \notin \varphi(Z_\varphi)$ and $|q - p| < \operatorname{dist}(p, \varphi(\partial D))$.

We can now extend the definition of the degree to the continuous $\varphi \in C(\bar{D})$, whose degree is then the degree of a sufficiently good C^1 approximation to φ.

DEFINITION 1.3. Suppose that $\varphi \in C(\bar{D})$ and $p \notin \varphi(\partial D)$. Then we define $\deg(\varphi, D, p)$ to be $\deg(\psi, D, p)$, where ψ is any function in $C^1(\bar{D})$ such that $\|\varphi - \psi\| < \operatorname{dist}(p, \varphi(\partial D))$, with $\|\cdot\|$ the norm in $C(\bar{D})$ given by $\|g\| = \sup\{|g(x)| : x \in \bar{D}\}$.

It should be noted that the usual procedure in proofs of degree-theoretic results is to first prove the result for the "nice" case of C^1 functions and non-crease points p, and then prove the general result by a process of approximation.

We note that Definition 1.3 yields the following theorem.

THEOREM 1.1. *To each continuous* $\varphi: \bar{D} \to \mathbb{R}^N$ *and each* $p \notin \varphi(\partial D)$ *there is associated an integer* $\deg(\varphi, D, p)$ *with the following properties.*

(P1) Invariance under homotopy. *If* φ_t *is a family of maps* $\varphi_t \in C(\bar{D})$ *depending continuously on* t *in uniform topology on* $\bar{D}$, *and if* $p \in \mathbb{R}^N$ *is such that* $p \notin \varphi_t(\partial D)$ *for each* $t \in [0,1]$, *then*

$$\deg(\varphi_0, D, p) = \deg(\varphi_1, D, p).$$

(P2) Dependence only on the boundary values. *As a consequence of* (P1), *we have: If* $\varphi|_{\partial D} = \psi|_{\partial D}$ *and* $p \notin \varphi(\partial D) = \psi(\partial D)$, *then* $\deg(\varphi, D, p) = \deg(\psi, D, p)$.

(P3) Continuity. *The function* $\deg(\varphi, D, p)$ *is continuous in* φ *in the uniform topology in the following sense: Given* φ *and* $p \notin \varphi(\partial D)$, *there exists a uniform neighborhood* U *of* φ *such that, if* $\psi \in U$, *then* $p \notin \psi(\partial D)$ *and* $\deg(\varphi, D, p) = \deg(\psi, D, p)$.

(P4) Decomposition of the domain. *If* $D = \bigcup_i^m D_i$ *with each* D_i *open, the family* $\{D_i\}$ *is disjoint, and* $\partial D_i \subset \partial D$, *then for each* $p \notin \varphi(\partial D)$ *we have*

$$\deg(\varphi, D, p) = \sum_{i=1}^{m} \deg(\varphi, D_i, p).$$

(P5) *If* $p \notin \varphi(\bar{D})$, *then* $\deg(\varphi, D, p) = 0$. *If* p *and* q *belong to the same component of* $\mathbb{R}^N - \varphi(\partial D)$, *then* $\deg(\varphi, D, p) = \deg(\varphi, D, q)$.

(P6) Excision property. *If* $p \notin \varphi(\partial D)$, $K \subset \bar{D}$, K *is closed, and* $p \notin \varphi(K)$, *then* $\deg(\varphi, D, p) = \deg(\varphi, \bar{D} \setminus K, p)$.

(P7) Borsuk theorem. *If* $0 \in D$, D *is symmetric about* 0, $\varphi: \bar{D} \to \mathbb{R}^N$ *is odd (i.e.* $\varphi(-x) = -\varphi(x)$ *for all* $x \in \bar{D}$), *and* $0 \notin \partial(\partial D)$, *then* $\deg(\varphi, D, 0)$ *is an odd number.*

(P8) Multiplication property. *Let* $\varphi \in C(\bar{D})$ *and let* M *be a bounded, open set containing* $\varphi(\bar{D})$. *Let* $\Delta = M \setminus \varphi(\partial D)$, *and suppose that the components of* Δ *are* Δ_i, $i = 1, 2, \ldots$. *If* $\psi \in C(\bar{M})$ *and* $p \notin \psi(\varphi(\partial D)) \cup \psi(\partial M)$, *then*

$$\deg(\psi \circ \varphi, D, p) = \sum_{\Delta_i} \deg(\psi, \Delta_i, p) \deg(\varphi, D, \Delta_i),$$

where the summation is known to be finite.

1.2. Some Applications

We now use properties (P1)–(P8) to obtain some fixed point theorems, surjectivity theorems, and other results which will be needed.

THEOREM 1.2. *Suppose that* $\varphi \in C(\bar{D})$ *and* $p \notin \varphi(\partial D)$. *If* $\deg(\varphi, D, p) \neq 0$, *then there is an* $x \in D$ *such that* $\varphi(x) = p$.

Proof. If $p \notin \varphi(\bar{D})$, take $\psi \in C^1(D)$ such that $\|\varphi - \psi\| < \operatorname{dist}(p, \varphi(\bar{D}))$. Then $p \notin \psi(\bar{D})$, whence $\deg(\psi, D, p) = 0$ (by definition). Thus we have $\deg(\varphi, D, p) = 0$, using Definition 1.3. Hence $p \in \varphi(D)$ if $\deg(\varphi, D, p) \neq 0$. □

THEOREM 1.3 (Brouwer fixed point theorem). *Let D be an open subset of* $\mathbb{R}^N$ *such that* $\bar{D}$ *is homeomorphic to a closed unit ball* $\bar{B}$. *If* $\varphi \in C(\bar{D})$ *and* $\varphi(\bar{D}) \subset \bar{D}$, *then* φ *has a fixed point in* $\bar{D}$.

Proof. Let $h: \bar{D} \to B$ be the homeomorphism. Let $\psi = h \circ \varphi \circ h^{-1}$; ψ maps $\bar{B}$ into itself and is continuous. Now if $\psi(y) = y$ with $y \in \bar{B}$, there is an $x \in \bar{D}$ such that $y = h(x)$ and $h \circ \varphi(x) = h(x)$. Because h is a homeomorphism and $\varphi(x) \in \bar{D}$, it follows that $\varphi(x) = x$. Thus, to prove the theorem we need only show that ψ has a fixed point in $\bar{B}$.

If $\psi(x_0) = x_0$ for some $x_0 \in \partial B$ then there is nothing further to prove. Hence we assume $\psi(x) \neq x$ for $x \in \partial B$. Consider the homotopy

$$H_t(x) = x - t\psi(x), \quad x \in \bar{B}, \; t \in [0,1].$$

It is clear that if $x \in \partial B$ and $0 \leq t < 1$, then $t\psi(x) \in B$; hence $H_t(x) \neq 0$ for $x \in \partial B$ and $t \in [0,1]$. Since, by hypothesis, $0 \notin H_1(\partial B)$, we deduce from (P1) that $\deg(I - \psi, B, 0) = \deg(I, B, 0) = 1$. □

REMARK 1.1. The Brouwer fixed point theorem is also true in the following form: A continuous map of a closed bounded convex set in $\mathbb{R}^N$ into itself has a fixed point. (See the notes and problems at the end of this chapter for an indication of the proof.)

THEOREM 1.4. *Let* $0 \in D$, *and let* $\varphi: \bar{D} \to \mathbb{R}^N$ *be a continuous map such that*

(LS) $\varphi(x) \neq \lambda x$ *for all* $x \in \partial D$ *and all* $\lambda > 1$.

Then φ *has a fixed point x in D; that is,* $x = \varphi(x)$ *for some* $x \in \bar{D}$.

Proof. Consider the homotopy $H(t,x) = x - t\varphi(x)$ for $x \in \bar{D}$ and $t \in [0,1]$. We claim that $H(t,x) \neq 0$ for $t \in [0,1]$ and $x \in \partial D$. If not, there would exist $t_0 \in [0,1]$ and $x_0 \in \partial D$ such that $H(t_0, x_0) = x_0 - t_0\varphi(x_0) = 0$.

If $t_0 = 0$ then $x_0 = 0$, contradicting the fact that $x_0 \in \partial D$ and $0 \in D$. If $t_0 \in (0,1)$, then $x_0 - t_0\varphi(x_0) = 0$ implies that $\varphi(x_0) = (1/t_0)x_0$ and $1/t_0 > 1$, in contradiction to (LS). If $t_0 = 1$, then φ has a fixed point $x_0 \in \partial D$ and there is nothing to prove. Thus we may assume that $H(t,x) \neq 0$ for $x \in \partial D$ and $t \in [0,1]$. It follows from this and (P1) that $\deg(I-\varphi, B, 0) = \deg(I, B, 0)$. But $\deg(I,B,0) = 1$, and therefore φ has a fixed point in $\bar{D}$. □

THEOREM 1.5. *Suppose $\varphi: \mathbb{R}^N \to \mathbb{R}^N$ is continuous and such that*

$$(\varphi(x), x)/\|x\| \to \infty \quad \text{as } \|x\| \to \infty, \tag{1.2}$$

where $(\cdot,\cdot)$ is the inner product on $\mathbb{R}^N$ and $\|\cdot\| = (\cdot,\cdot)^{1/2}$ its Euclidean norm. Then φ is onto $\mathbb{R}^N$; that is, for each $p \in \mathbb{R}^N$ the equation $\varphi(x) = p$ has a solution in $\mathbb{R}^N$.

Proof. We may suppose without loss of generality that $p = 0$. Now, for some $r > 0$ we have $(\varphi(x), x) \geq 0$ if $\|x\| = r$. Suppose $\varphi(x) \neq 0$ for $\|x\| = r$; otherwise, we are finished. Then $(\varphi(x), x) \geq 0$ implies that

$$\varphi(x) + \lambda x \neq 0 \quad \text{for all } \lambda \geq 0 \text{ and } \|x\| = r.$$

Consider the homotopy $H(t,x) = t\varphi(x) + (1-t)x$ for $x \in \partial B(0,r)$ and $t \in [0,1]$. By hypothesis, $H(t,x) \neq 0$ for $x \in \partial D$ and $t \in [0,1]$. Hence

$$\deg(\varphi, B, 0) = \deg(I, B, 0) = 1,$$

and thus $\varphi(x) = 0$ has a solution in $B(0,r)$. □

When we come to define degree in infinite dimensional spaces X or (X, Y) in Chapter 2, we shall need to apply the theory developed here in finite dimensional spaces other than $\mathbb{R}^N$.

Suppose X is a real vector space of dimension N. By choosing a basis in X we can identify X with $\mathbb{R}^N$. This should allow us to define $\deg(\varphi, D, p)$ as was done for $\mathbb{R}^N$; of course, the only important thing is to see what happens after a change of basis. The answer is that the degrees are equal in all cases. More precisely, given a basis $B = (b_1, \ldots, b_N)$, we shall for the moment denote by $\deg_B(\varphi, D, p)$ the degree computed with respect to B. Then we have the next proposition.

PROPOSITION 1.2. *For every pair of bases A and B,*

$$\deg_A(\varphi, D, p) = \deg_B(\varphi, D, p),$$

whenever the expressions make sense.

The proposition is proved by using C^1 maps and the invariance of the sign of the Jacobian under the change of bases.

PROPOSITION 1.3. *The degree theory outlined here is also valid when X and Y are finite dimensional spaces, $\varphi: \bar{D} \subset X \to Y$ is continuous, and p in Y is such that $p \notin \varphi(\partial D)$, provided X and Y are oriented and have the same dimension.*

The next result is sometimes referred to as the "invariance of a normal."

THEOREM 1.6. *Suppose that $0 \in D \subset \mathbb{R}^N$ and that N is odd. If $\varphi \in C(\bar{D})$ and $0 \notin \varphi(\partial D)$, then there are $y \in \partial D$ and $\lambda \neq 0$ such that $\varphi(y) = \lambda y$.*

Proof. Define the homotopies H_t and K_t by

$$H_t(x) = (1-t)\varphi(x) + tx,$$

$$K_t(x) = (1-t)\varphi(x) - tx,$$

where $x \in \bar{D}$ and $0 \le t \le 1$. If no $y \in \partial D$ and $\lambda \neq 0$ can be found to satisfy $\varphi(y) = \lambda y$, then $H_t(x) \neq 0$ and $K_t(x) \neq 0$ for $x \in \partial D$ and $0 < t \le 1$. Since $0 \notin \varphi(\partial D)$, $H_0(x)$ and $K_0(x)$ are also nonzero for $x \in \partial D$. Hence (P1) of Theorem 1.1, applied to H_t and K_t, respectively yields

$$\deg(\varphi, D, 0) = \deg(I, D, 0),$$

$$\deg(\varphi, D, 0) = \deg(-I, D, 0).$$

Now $\deg(I, D, 0) = 1$ and it is seen from the definition of the degree that $\deg(-I, D, 0) = (-1)^N$. We thus have $1 = (-1)^N$, whence N is even, contrary to our hypothesis. □

The condition that N be odd is necessary for Theorem 1.6 to be true. A counterexample is given by the map of the unit disc in $\mathbb{R}^2$ given in polar coordinates by $(r, \theta) \mapsto (r, \theta + r)$.

THEOREM 1.7 (antipode theorem). *Let D be symmetric about $0 \in D \subset \mathbb{R}^N$. If $\varphi: \bar{D} \to \mathbb{R}^N$ is a C function, $0 \notin \varphi(\partial D)$, and*

$$\frac{\varphi(x)}{|\varphi(x)|} \neq \frac{\varphi(-x)}{|\varphi(-x)|} \quad \text{for all } x \in \partial D, \tag{1.3}$$

then $\deg(\varphi, D, 0)$ *is an odd number.*