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Locales

A topological space is built up from points and open subsets. But in some sense, open subsets are much more essential than points. Indeed, knowing the points of the space does not give any information on the topology. On the other hand, the lattice of open subsets very often characterizes at the same time the set of points. For example in a Hausdorff space, the points of the space can be characterized as the atoms of the lattice of closed subsets, which is just the dual of the lattice of open subsets.

A locale is just a lattice which mimics the properties of the lattice of open subsets of a topological space. Locales appear very naturally when studying sheaves (see chapters 2, 3) and, even when studying sheaves on a topological space, many constructions lead to the consideration of locales which are no longer isomorphic to the lattice of open subsets of a space.

But besides generalizing nicely the notion of topological space, locales are important for a completely different reason: they satisfy all the axioms of intuitionistic propositional calculus, i.e., roughly speaking, the classical propositional calculus without the law of excluded middle. This last remark will turn out to grow in importance through the remaining chapters of this book. We shall start by making more precise this reference to intuitionistic logic.

1.1 The intuitionistic propositional calculus

Let us consider a mathematical theory, say the theory of groups, and all the formulas we can write in this theory, like

\[ \forall x \ \exists y \ x + y = 0, \]

\[ \forall x \ \forall y \ x + y = 0. \]
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Some of the formulas are true (like the first one), some of the formulas are false (like the second one). To prove the validity of a formula, one uses of course the axioms of the theory of groups, but also the axioms and deduction rules of propositional and predicate calculus. The propositional calculus is, roughly speaking, the theory which studies the various consequences one can infer from the validity of some formulas, by combining them using the logical connectors $\land$ (and), $\lor$ (or), $\Rightarrow$ (implies), $\neg$ (not). The predicate calculus takes additionally into account the two quantifiers $\exists$ (there exists) and $\forall$ (for all). Clearly, the consequences we can infer from some data depend on the assumptions we accept for our propositional or predicate calculus. First of all there are axioms, which declare that some types of formulas are necessarily “true”; we write $\vdash \varphi$ to indicate the truth of $\varphi$. Next there are deduction rules, which assert that given some true formulas, some derived formula is true as well.

Definition 1.1.1 The intuitionistic propositional calculus is the one having for axioms:

$(PC1) \vdash \varphi \Rightarrow (\psi \Rightarrow \varphi)$;
$(PC2) \vdash (\varphi \Rightarrow (\psi \Rightarrow \theta)) \Rightarrow ((\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \theta))$;
$(PC3) \vdash \varphi \Rightarrow (\psi \Rightarrow (\varphi \land \psi))$;
$(PC4) \vdash (\varphi \land \psi) \Rightarrow \varphi$
$(PC5) \vdash (\varphi \land \psi) \Rightarrow \psi$
$(PC6) \vdash \varphi \Rightarrow (\varphi \lor \psi)$
$(PC7) \vdash \psi \Rightarrow (\varphi \lor \psi)$
$(PC8) \vdash (\varphi \Rightarrow \theta) \Rightarrow ((\psi \Rightarrow \theta) \Rightarrow (\varphi \lor \psi \Rightarrow \theta))$
$(PC9) \vdash (\varphi \Rightarrow \psi) \Rightarrow ((\varphi \Rightarrow \neg \psi) \Rightarrow \neg \varphi)$
$(PC10) \vdash \neg \varphi \Rightarrow (\varphi \Rightarrow \psi)$;

and for rule of deduction, the modus ponens:

if $\vdash \varphi$ and $\vdash \varphi \Rightarrow \psi$, then $\vdash \psi$.

In this definition, $\varphi, \psi, \theta$ are arbitrary formulas.

Definition 1.1.2 The classical propositional calculus is the one obtained from the intuitionistic propositional calculus by adding the axiom

$\vdash \varphi \lor \neg \varphi$

(the so-called “law of the excluded middle”).
1.1 The intuitionistic propositional calculus

Lemma 1.1.3 In intuitionistic propositional calculus, putting
\[
\varphi \leq \psi \iff \vdash \varphi \Rightarrow \psi
\]
provides the set \( \mathcal{F} \) of formulas with the structure of a preordered set. This preordered set is finitely complete and cocomplete and for each formula \( \varphi \), the functor
\[
\neg \land \varphi : \mathcal{F} \to \mathcal{F}; \quad \theta \mapsto \theta \land \varphi
\]
admits the functor
\[
\varphi \Rightarrow \neg : \mathcal{F} \to \mathcal{F}; \quad \psi \mapsto \varphi \Rightarrow \psi
\]
as a right adjoint.

Proof First of all, let us prove that for a formula \( \varphi \), \( \varphi \leq \varphi \), i.e. \( \vdash \varphi \Rightarrow \varphi \).
By (PC1), (PC2) and the modus ponens we get
\[
\vdash (\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \varphi).
\]
Replacing \( \psi \) by \( \varphi \Rightarrow \varphi \), applying (PC1) and the modus ponens, we get
\[
\vdash \varphi \Rightarrow \varphi.
\]
To get a preorder, we must still prove that \( \varphi \leq \psi \) and \( \psi \leq \theta \) imply \( \varphi \leq \theta \), i.e. \( \vdash \varphi \Rightarrow \psi \) and \( \vdash \psi \Rightarrow \theta \) imply \( \vdash \varphi \Rightarrow \theta \). By (PC1) one has
\[
\vdash (\theta \Rightarrow \theta) \Rightarrow (\varphi \Rightarrow (\psi \Rightarrow \theta)).
\]
Applying the modus ponens this yields \( \vdash \varphi \Rightarrow (\varphi \Rightarrow \theta) \). Applying (PC2) and the modus ponens twice yields \( \vdash \varphi \Rightarrow \theta \).

( PC4) and ( PC5) yield \( \varphi \land \psi \leq \varphi \), \( \varphi \land \psi \leq \psi \). To prove that \( \varphi \land \psi \) is the infimum of \( \varphi \), \( \psi \), choose \( \theta \) such that \( \theta \leq \varphi \), \( \theta \leq \psi \). So \( \vdash \theta \Rightarrow \varphi \) and \( \vdash \theta \Rightarrow \psi \), from which we must deduce \( \theta \leq \varphi \land \psi \), i.e. \( \vdash \theta \Rightarrow \varphi \land \psi \). From \( \vdash \theta \Rightarrow \varphi \) and (PC3) we deduce by transitivity that
\[
\vdash \theta \Rightarrow (\psi \Rightarrow (\varphi \land \psi)).
\]
In (PC2), let us choose \( \varphi \) to be \( \theta \) and \( \theta \) to be \( \varphi \land \psi \); applying the modus ponens we get
\[
\vdash (\theta \Rightarrow \psi) \Rightarrow (\theta \Rightarrow (\varphi \land \psi)).
\]
A last application of the modus ponens yields \( \vdash \theta \Rightarrow (\varphi \land \psi) \) as required.
In particular \( \varphi \land \psi \) and \( \psi \land \varphi \) are isomorphic, i.e. \( \varphi \land \psi \leq \varphi \land \psi \) and \( \psi \land \varphi \leq \varphi \land \psi \).

( PC6) and ( PC7) yield \( \varphi \leq \varphi \lor \psi \), \( \psi \leq \varphi \lor \psi \). To prove that \( \varphi \lor \psi \) is the supremum of \( \varphi \), \( \psi \), choose \( \theta \) such that \( \varphi \leq \theta \), \( \psi \leq \theta \). So \( \vdash \varphi \Rightarrow \theta \) and \( \vdash \psi \Rightarrow \theta \), from which we must deduce \( \varphi \lor \psi \leq \theta \), i.e. \( \vdash (\varphi \lor \psi) \Rightarrow \theta \). This is immediate from (PC8) and a double application of the modus ponens.
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In particular again, \( \varphi \lor \psi \) and \( \psi \lor \varphi \) are isomorphic, i.e. \( \varphi \lor \psi \leq \psi \lor \varphi \) and \( \psi \lor \varphi \leq \varphi \lor \psi \).

To get the terminal object, consider an arbitrary true formula \( \varphi \), i.e. \( \vdash \varphi \). (For example choose \( \varphi \) to be \( \theta \Rightarrow \theta \) for an arbitrary formula \( \theta \).) The formula \( \varphi \) will be terminal if for every formula \( \psi \), \( \psi \leq \varphi \), that is \( \vdash \psi \Rightarrow \varphi \). This follows immediately from (PC1) and the modest ponens. In particular, if \( \varphi, \psi \) are true formulas, they are isomorphic, i.e. \( \varphi \leq \psi \) and \( \psi \leq \varphi \).

To get the initial object, consider again a true formula \( \varphi \), i.e. \( \vdash \varphi \). Then \( \neg \varphi \) will be initial if for every formula \( \psi \), \( \neg \varphi \leq \psi \), i.e. \( \vdash \neg \varphi \Rightarrow \psi \). Via (PC10), with \( \varphi \) replaced by \( \neg \varphi \), and the modest ponens, this reduces to proving \( \vdash \neg \neg \varphi \). By (PC9) we have

\[
\vdash (\neg \varphi \Rightarrow \varphi) \Rightarrow ((\neg \varphi \Rightarrow \neg \varphi) \Rightarrow \neg \neg \varphi).
\]

Since \( \varphi \) is true, \( \varphi \) is terminal and \( \vdash \neg \varphi \Rightarrow \varphi \). A double application of the modest ponens yields \( \vdash \neg \neg \varphi \).

It remains to verify the stated adjunction. First of all, we have to check the functoriality of the two constructions. For \( \neg \land \varphi \) this is obvious, since this reduces to choosing the infimum with \( \varphi \). For \( \varphi \Rightarrow \neg \), we must prove that \( \psi \leq \theta \) implies \( (\varphi \Rightarrow \psi) \leq (\varphi \Rightarrow \theta) \). In other words, if \( \vdash \psi \Rightarrow \theta \), then

\[
\vdash (\varphi \Rightarrow \psi) \Rightarrow (\varphi \Rightarrow \theta).
\]

But if \( \vdash \psi \Rightarrow \theta \), \( \psi \Rightarrow \theta \) is terminal so that \( \vdash \varphi \Rightarrow (\psi \Rightarrow \theta) \). The functoriality of \( \varphi \Rightarrow \neg \) follows then from (PC2) and the modest ponens.

To prove the required adjunction, we apply condition (2) of 3.1.5, volume 1. Given formulas \( \theta, \varphi, \psi \), (PC3) immediately implies

\[
\theta \leq (\varphi \Rightarrow (\theta \land \varphi)).
\]

To prove the second condition, observe first that (PC4) yields

\[
\vdash ((\varphi \Rightarrow \psi) \land \varphi) \Rightarrow (\varphi \Rightarrow \psi).
\]

Applying (PC2) and the modest ponens we get

\[
\vdash ((\varphi \Rightarrow \psi) \land \varphi) \Rightarrow ( ((\varphi \Rightarrow \psi) \land \varphi) \Rightarrow \psi).
\]

Now applying (PC4) and the modest ponens again yields

\[
\vdash ((\varphi \Rightarrow \psi) \land \varphi) \Rightarrow \psi
\]

which is the required inequality.  \( \Box \)
1.2 Heyting algebras

Lemma 1.1.4 In the intuitionistic propositional calculus, with the notation of 1.1.3, for every formula \( \varphi \) the mapping

\[
- \Rightarrow \psi : \mathcal{F} \rightarrow \mathcal{F}, \quad \varphi \mapsto \varphi \Rightarrow \psi
\]

is a contravariant functor. When \( \psi = \perp \) is the initial object of \( \mathcal{F} \), this functor is isomorphic to

\[
\text{false} : \mathcal{F} \rightarrow \mathcal{F}, \quad \varphi \mapsto \neg \varphi.
\]

Proof Suppose \( \theta \leq \varphi \); we must prove that \( (\varphi \Rightarrow \psi) \leq (\theta \Rightarrow \psi) \). By adjunction (see 1.1.3), this is equivalent to \( (\varphi \Rightarrow \psi) \land \theta \leq \psi \). And indeed

\[
(\varphi \Rightarrow \psi) \land \theta \leq (\varphi \Rightarrow \psi) \land \varphi \leq \psi
\]

by 1.1.3 and 3.1.5(2), volume 1.

Next by (PC10) we have immediately \( \neg \varphi \leq (\varphi \Rightarrow \perp) \). Conversely we know that \( \perp = \neg \top \), where \( \top \) is any true formula (see proof of 1.1.3). So applying (PC9) we get

\[
\vdash (\varphi \Rightarrow \top) \Rightarrow ((\varphi \Rightarrow \neg \top) \Rightarrow \neg \varphi).
\]

But \( \vdash \varphi \Rightarrow \top \) since \( \top \) is terminal, thus applying the modus ponens yields

\[
(\varphi \Rightarrow \perp) \leq \neg \varphi.
\]

Again with the notation of 1.1.3, it is a common practice to consider the quotient of the preordered set \( \mathcal{F} \) of formulas by the equivalence relation identifying two formulas \( \varphi, \psi \) when they are isomorphic, i.e. when

\[
\vdash \varphi \Rightarrow \psi \quad \text{and} \quad \vdash \psi \Rightarrow \varphi.
\]

Writing \([\varphi]\) for the equivalence class of the formula \( \varphi \), the quotient is now an actual lattice in which each functor \(- \land [\varphi]\) has a right adjoint. This is what we shall call a Heyting algebra. In the case of classical propositional calculus, it is well known that this Heyting algebra is in fact a boolean algebra.

1.2 Heyting algebras

As suggested at the end of the previous section, we state

Definition 1.2.1 A Heyting algebra \( \mathcal{H} \) is a lattice, with top and bottom elements, in which for every element \( b \in \mathcal{H} \), the functor

\[
- \land b : \mathcal{H} \rightarrow \mathcal{H}, \quad a \mapsto a \land b,
\]

has a right adjoint, which we shall denote

\[
b \Rightarrow -: \mathcal{H} \Rightarrow \mathcal{H}, \quad c \mapsto b \Rightarrow c.
\]

We shall write 1 for the top element and 0 for the bottom element.
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The adjunction indicated thus reduces to the relation

\[ a \land b \leq c \quad \text{iff} \quad a \leq b \Rightarrow c \]

for arbitrary elements \(a, b, c\).

**Proposition 1.2.2** A Heyting algebra is necessarily a distributive lattice.

**Proof** Since the functor \(- \land b\) has a right adjoint, it preserves colimits (see 3.2.2, volume 1). Therefore

\[ (a \lor a') \land b = (a \land b) \lor (a' \land b). \]

We recall that this implies the other distributivity law

\[ (a \land a') \lor b = (a \lor b) \land (a' \lor b). \]

Indeed

\[
\begin{align*}
(a \lor b) \land (a' \lor b) &= (a \land (a' \lor b)) \lor (b \lor (a' \land b)) \\
&= (a \land a') \lor (a \land b) \lor (b \land a') \lor (b \land b) \\
&= (a \land a') \lor b.
\end{align*}
\]

**Proposition 1.2.3** In a Heyting algebra, the following relations hold:

1. \(a \leq b\) iff \((a \Rightarrow b) = 1\);
2. \(a = (1 \Rightarrow a)\);
3. \(a \Rightarrow (b \land c) = (a \Rightarrow b) \land (a \Rightarrow c)\);

for arbitrary elements \(a, b, c\).

**Proof** \((a \Rightarrow b) = 1\) is equivalent to \(1 \leq (a \Rightarrow b)\), i.e. by adjunction to \(1 \land a \leq b\). In the same way \(a \leq (1 \Rightarrow a)\) reduces by adjunction to \(a \land 1 \leq a\). On the other hand \((1 \Rightarrow a) \leq (1 \Rightarrow a)\) yields by adjunction \((1 \Rightarrow a) \land 1 \leq a\), which is the converse inequality. The third relation holds because \(a \Rightarrow -\), having a left adjoint, preserves infima (see 3.2.2, volume 1).

**Proposition 1.2.4** In a Heyting algebra, \(b \Rightarrow c\) is the greatest element such that \(b \land (b \Rightarrow c) \leq c\), i.e.

1. \(b \Rightarrow c = \bigvee \{a \mid a \land b \leq c\}\);
2. \((b \Rightarrow c) \land b \leq c\).

**Proof** \((b \Rightarrow c) \land b \leq c\) reduces to \((b \Rightarrow c) \leq (b \Rightarrow c)\) by adjunction.

Next, if \(a \land b \leq c\), again by adjunction \(a \leq b \Rightarrow c\).
1.2 Heyting algebras

**Proposition 1.2.5** In a Heyting algebra $\mathcal{H}$, for every element $c$ the mapping

$$- \Rightarrow c: \mathcal{H} \longrightarrow \mathcal{H}, \quad b \mapsto b \Rightarrow c,$$

is a contravariant functor.

**Proof** If $b \leq b'$, we must prove that $(b' \Rightarrow c) \leq (b \Rightarrow c)$. Indeed by 1.2.4

$$(b' \Rightarrow c) \land b \leq (b' \Rightarrow c) \land b' \leq c,$$

from which the result follows by adjunction.

**Proposition 1.2.6** In a Heyting algebra, putting $\neg b = (b \Rightarrow 0)$ yields the greatest element such that $b \land \neg b = 0$, i.e.

1. $\neg b = \bigvee \{a | a \land b = 0\}$;
2. $\neg b \land b = 0$.

The element $\neg b$ is called the pseudo-complement of $b$.

**Proof** This is just 1.2.4 with $c = 0$.

**Proposition 1.2.7** In a Heyting algebra, the following conditions hold:

1. $\neg 0 = 1$, $\neg 1 = 0$;
2. $a \leq b$ implies $\neg b \leq \neg a$;
3. $\neg a = \neg \neg \neg a$;
4. $\neg(a \lor b) = \neg a \land \neg b$;
5. $\neg a \lor b \leq a \Rightarrow b$;

for all elements $a, b$.

**Proof** $\neg 0 = 1$ is equivalent to $1 \leq (0 \Rightarrow 0)$, i.e. $1 \land 0 \leq 0$. On the other hand $\neg 1 = \neg 1 \land 1 = 0$. The second statement is just 1.2.5 in the case $c = 0$.

By adjunction $a \leq \neg \neg a$ is equivalent to $a \land \neg a = 0$, which holds by 1.2.6. Applying the second statement, this yields $\neg \neg \neg a \leq \neg a$. On the other hand $\neg a \leq \neg \neg \neg a$ is equivalent by adjunction to $\neg a \land \neg \neg a = 0$, which holds again by 1.2.6.

Since $a \leq a \lor b$, $b \leq a \lor b$, the second statement immediately implies $\neg(a \lor b) \leq \neg a \land \neg b$. The converse inequality reduces by adjunction to

$$\neg a \land \neg b \land (a \lor b) = 0.$$

Applying 1.2.2 one has

$$\neg a \land \neg b \land (a \lor b) = (\neg a \land \neg b \land a) \lor (\neg a \land \neg b \land b)$$

$$= (0 \land \neg b) \lor (\neg a \land 0)$$

$$= 0.$$
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Finally, the last statement follows by adjunction from

\[(\neg a \lor b) \land a = (\neg a \land a) \lor (b \land a) = 0 \lor (b \land a) \leq b.\]

\[\square\]

**Proposition 1.2.8** Let \( \mathcal{H} \) be a Heyting algebra. The double negation mapping

\[\neg \neg \colon \mathcal{H} \longrightarrow \mathcal{H}, \quad a \mapsto \neg \neg a,\]

satisfies the following conditions:

1. \( a \leq b \) implies \( \neg \neg a \leq \neg \neg b; \)
2. \( a \leq \neg \neg a; \)
3. \( \neg 0 = 0, \neg 1 = 1; \)
4. \( \neg (\neg \neg a) = \neg \neg a; \)
5. \( \neg \neg (a \land b) = \neg \neg a \land \neg \neg b; \)
6. \( \neg \neg (a \Rightarrow b) = (\neg \neg a \Rightarrow \neg \neg b). \)

**Proof** Statements (1), (2), (3), (4) follow immediately from 1.2.7 or its proof.

To prove (5), observe first that \( a \land b \leq a \) and \( a \land b \leq b \) yield already, by (1), \( \neg \neg (a \land b) \leq \neg \neg a \land \neg \neg b. \) To prove the converse, observe that

\[\neg (a \land b) \land a \land b = 0 \implies \neg (a \land b) \land a \leq \neg b = \neg \neg b\]

implies

\[\neg (a \land b) \land a \land \neg \neg b = 0\]

implies

\[\neg (a \land b) \land \neg \neg b \leq \neg a = \neg \neg \neg a\]

implies

\[\neg (a \land b) \land \neg \neg b \land \neg \neg a = 0\]

implies

\[\neg \neg a \land \neg \neg b \leq \neg (a \land b).\]

For the last statement, one has immediately, applying (5) and (1)

\[\neg \neg (a \Rightarrow b) \land \neg \neg a = \neg \neg ((a \Rightarrow b) \land a) \leq \neg \neg b,\]

from which \( \neg \neg (a \Rightarrow b) \leq (\neg \neg a \Rightarrow \neg \neg b). \) Conversely, given any element \( c \in \mathcal{H} \)

\[\neg \neg a \land c \leq \neg \neg b \implies \neg \neg a \land c \land \neg \neg b = 0\]

implies

\[\neg (\neg a \lor b) \land c = 0\]

implies

\[\neg (a \Rightarrow b) \land c = 0\]

implies

\[c \leq \neg \neg (a \Rightarrow b),\]

see 1.2.7. Putting \( c = \neg \neg a \land \neg \neg b \) yields the required result. \[\square\]

In a Heyting algebra \( \mathcal{H} \), the two relations

\[\neg (a \lor b) = \neg a \land \neg b, \quad \neg (a \land b) = \neg a \lor \neg b\]
1.2 Heyting algebras

are referred to as the two De Morgan laws. The first one holds in every Heyting algebra (see 1.2.7), but the second one does not in general (see 1.3.4.c for a counterexample).

Proposition 1.2.9 For a Heyting algebra $\mathcal{H}$, the following conditions are equivalent:

1. $\mathcal{H}$ satisfies the two De Morgan laws;
2. $\forall a, b \in \mathcal{H} \quad \lnot(a \land b) = \lnot a \lor \lnot b$;
3. $\forall a \in \mathcal{H} \quad \neg \neg a = 1$;
4. $\forall a, b \in \mathcal{H} \quad \neg \neg(a \lor b) = \neg \neg a \lor \neg \neg b$.

Proof (1) is equivalent to (2) by 1.2.7.(4). (2) immediately implies (3), putting $b = \neg a$.

Let us prove that (3) implies (4).

\[
\neg \neg(a \lor b) = \neg \neg(a \lor b) \land \neg \neg a \\
= (\neg \neg(a \lor b) \land \neg \neg a) \lor (\neg \neg(a \lor b) \land \neg \neg a) \\
= (\neg \neg(a \lor b) \land \neg \neg a) \lor \neg \neg a \\
= \neg \neg((a \lor b) \land \neg a) \lor \neg \neg a \\
= \neg \neg((a \land \neg a) \lor (b \land \neg a)) \lor \neg \neg a \\
= \neg \neg(b \land \neg a) \lor \neg \neg a \\
= (\neg \neg b \land \neg \neg a) \lor \neg \neg a \\
= (\neg \neg b \lor \neg \neg a) \land (\neg \neg a \lor \neg \neg a) \\
= \neg \neg b \lor \neg \neg a.
\]

Finally we prove that (4) implies (2):

\[
\neg(a \land b) = \neg \neg(a \land b) \\
= \neg \neg \neg a \land \neg \neg b \\
= \neg \neg(a \land \neg b) \\
= \neg \neg a \lor \neg \neg b \\
= a \lor \neg b.
\]

Next we consider the even more special case of a boolean algebra.

Proposition 1.2.10 Every boolean algebra is a Heyting algebra satisfying the two De Morgan laws.

Proof Writing $\neg b$ for the complement of the element $b$, one thus has $b \land \neg b = 0$ and $b \lor \neg b = 1$. Let us prove that $b \Rightarrow c$ exists, for two
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arbitrary elements $b, c$, and is given by

$$b \Rightarrow c = \neg b \lor c.$$  

For every element $a$, we must prove that

$$a \land b \leq c \iff a \leq \neg b \lor c.$$  

Since a boolean algebra is in particular a distributive lattice, $a \land b \leq c$ implies

$$a = a \land (b \lor \neg b) = (a \land b) \lor (a \land \neg b) \leq c \lor \neg b.$$  

Conversely from $a \leq \neg b \lor c$ we deduce

$$a \land b \leq (\neg b \lor c) \land b = (\neg b \land b) \lor (c \land b) = c \land b \leq c.$$  

So every boolean algebra is a Heyting algebra.

Next observe that

$$b \Rightarrow 0 = \neg b \lor 0 = \neg b$$

which proves that $\neg b$ is the pseudo-complement of $b$, in the sense of 1.2.6. Therefore the relation $b \lor \neg b = 1$ implies, putting $b = \neg a$, that $\neg a \lor \neg \neg a = 1$. So a boolean algebra satisfies the two De Morgan laws (see 1.2.9).

Proposition 1.2.11 For a Heyting algebra $\mathcal{H}$, the following conditions are equivalent:

1. $\mathcal{H}$ is a boolean algebra;
2. $\forall a \in \mathcal{H} \ a \lor \neg a = 1$;
3. $\forall a \in \mathcal{H} \ \neg \neg a = a$.

Proof If $\mathcal{H}$ is a boolean algebra, we have observed in proving 1.2.10 that the complement of an element is also its pseudo-complement. So (1) implies (2).

Suppose $a \lor \neg a = 1$ for every element of a Heyting algebra. By 1.2.8, we already have $a \leq \neg \neg a$. Therefore

$$\neg \neg a = \neg \neg a \land (a \lor \neg a) = (\neg \neg a \land a) \lor (\neg \neg a \land \neg a) = a \lor 0 = a.$$  

Next suppose $\neg \neg a = a$ for every element $a$ of a Heyting algebra. Thus $\mathcal{H}$ is a distributive lattice (see 1.2.2) and it remains to prove that $\neg a$ is the complement of $a$. By 1.2.6, $a \land \neg a = 0$. Next,

$$a \lor \neg a = \neg \neg (a \lor \neg a) = \neg (\neg a \land \neg a) = \neg 0 = 1$$

by 1.2.7.

In view of condition 1.2.11.(3), let us make the following definition: