ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

Edited by G.-C. ROTA
Volume 51

Handbook of Categorical Algebra 2
à René Lavendhomme,
mon maître
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Preface to volume 2

This second volume of the *Handbook of categorical algebra* presents a selection of well-known specialized topics in category theory, with the exception of toposes which find their natural place in volume 3.

The first great achievement of category theory has certainly been the theory of abelian categories: these play an important role in homology and provide the correct setting for studying problems related to exact sequences. Entire books are devoted to abelian categories; in the first chapter of this volume we have selected some topics on abelian categories which appear to remain highly relevant in to-day’s research in general category theory. The chapter starts by establishing the key “exactness” properties of limits and colimits in an abelian category, those properties being closely related to the existence of an “additive structure” on the sets of morphisms. The notion of exact sequence is then introduced together with the technique of “diagram chasing”, used to prove the fundamental diagram lemmas. “Diagram chasing” is a technique for proving exactness properties in an abelian category, just by proving them in the categories of modules, where actual elements can be used. This is best achieved by applying the famous “embedding theorem” which asserts that every small abelian category can be fully and exactly embedded in a category of modules. This very difficult theorem is proved at the end of the chapter, using a Lubkin completion technique which turns out to be applicable in many other categorical situations. For those who do not want to enter these difficult matters, we give a very elementary “diagram chasing metatheorem” which is good enough for many purposes. We also study the localizations of abelian categories and their relations with torsion theories and universal closure operations.

Regular and exact categories are in a way those categories which re-capture some essential exactness properties of abelian categories, but
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without any requirement or implication of additivity. As examples, one gets most “algebra-like” categories. Regular categories provide the correct setting for developing the theory of relations, in particular equivalence relations. Making the technique even harder, the proof we have given of the embedding theorem for abelian categories can be adapted to produce a full exact embedding of every small category in a topos of presheaves, i.e. in a category \( \text{Fun}(\mathcal{D}^{\text{op}}, \text{Set}) \) for some small category \( \mathcal{D} \). We have preferred the much easier full exact embedding theorem in a topos of sheaves (see chapter 3, volume 3), which provides almost as good a “diagram chasing” metatheorem.

The next three chapters are devoted to various categorical approaches to the notion of “model of an algebraic theory”. In chapter 3, we are interested in those cases where the algebraic structure is given by finitary everywhere defined operations, the axioms being expressed by equalities. The corresponding categories of models can be presented as categories of set-valued finite product preserving functors. We study the completeness and exactness properties of these “algebraic categories” and pay special attention to the case of free models. We prove a characterization theorem for algebraic categories and conclude the chapter with some special topics: commutative theories, tensor product of theories, and Morita equivalent theories.

The notion of “monad on a category” formalizes intuitively the idea of a theory defined by “operations of arbitrary arities”. We show the close relations of this with the notion of adjoint functors and study the completeness and exactness properties of “monadic categories”; we prove also a corresponding characterization theorem. We pay some additional attention to the case of monads with finite rank (intuitively: the possible arities of operations are now bounded by some cardinal) and we conclude the chapter by exhibiting some relations with descent theory for modules.

In chapter 3, algebraic theories were defined using finite products. In chapter 5, we investigate those theories which can be defined by a “sketch”, that is a set of small limits and colimits. The corresponding categories of models are the “accessible categories”: they turn out to be exactly the categories of set-valued \( \alpha \)-flat functors, for some regular cardinal \( \alpha \). When just limit cones are used, the corresponding accessible categories are “locally presentable” and coincide with the categories of set-valued \( \alpha \)-left-exact functors, for some regular cardinal \( \alpha \).

Chapter 6 introduces some fundamental notions and results of enriched category theory. We are interested here in the case where the
categories involved have an additional structure on their sets $\mathcal{C}(A, B)$ of morphisms; for example the categories of modules on a ring have abelian groups of morphisms: they are “enriched” in the category of abelian groups. We limit our investigations to the basic notions and questions concerning enriched limits, enriched adjunctions and enriched Kan extensions. Cartesian closed categories – i.e. those categories in which the cartesian product with every object has a right adjoint – constitute an important example of categories in which to enrich category theory: in particular, all toposes (see chapter 5, volume 3) are cartesian closed.

Unfortunately the category of topological spaces is not cartesian closed, i.e. given topological spaces $Y, Z$, there is no way to provide the set $\mathcal{C}(Y, Z)$ of continuous functions with a topology, in such a way that a mapping

$$X \longrightarrow \mathcal{C}(Y, Z)$$

is continuous if and only if the corresponding mapping

$$X \times Y \longrightarrow Z$$

is continuous, for every other space $X$. We pay some attention to “exponentiable spaces $Y$” (those for which the previous problem has a solution) and show that restricting one’s attention to compactly generated spaces yields a cartesian closed category of topological spaces. Finally we introduce topological functors, i.e. those functors which satisfy axiomatically the conditions for the existence of topological-like initial structures.

The last chapter of this volume is devoted to the theory of fibred categories “à la Bénabou”. Fibred categories formalize the idea of “families of objects and morphisms” indexed by an object in a base category. We study first the corresponding fibred notions of adjunction and completeness. We then pay special attention to “locally small” fibrations, which are those for which the formal “families of morphisms” are represented by objects in the base category. We conclude with the very crucial notion of “definability” which exhibits those classes of devices in a fibration which can be represented by objects in the base category.
Introduction to this handbook

My concern in writing the three volumes of this *Handbook of categorical algebra* has been to propose a directly accessible account of what – in my opinion – a Ph.D. student should ideally know of category theory before starting research on one precise topic in this domain. Of course, there are already many good books on category theory: general accounts of the state of the art as it was in the late sixties, or specialized books on more specific recent topics. If you add to this several famous original papers not covered by any book and some important but never published works, you get a mass of material which gives probably a deeper insight in the field than this *Handbook* can do. But the great number and the diversity of those excellent sources just act to convince me that an integrated presentation of the most relevant aspects of them remains a useful service to the mathematical community. This is the objective of these three volumes.

The first volume presents those basic aspects of category theory which are present as such in almost every topic of categorical algebra. This includes the general theory of limits, adjoint functors and Kan extensions, but also quite sophisticated methods (like categories of fractions or orthogonal subcategories) for constructing adjoint functors. Special attention is also devoted to some refinements of the standard notions, like Cauchy completeness, flat functors, distributors, 2-categories, bicategories, lax-functors, and so on.

The second volume presents a selection of the most famous classes of “structured categories”, with the exception of toposes which appear in volume 3. The first historical example is that of abelian categories, which we follow by its natural non-additive generalizations: the regular and exact categories. Next we study various approaches to “categories of models of a theory”: algebraic categories, monadic categories, locally
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presentable and accessible categories. We introduce also enriched category theory and devote some attention to topological categories. The volume ends with the theory of fibred categories “à la Bénabou”.

The third volume is entirely devoted to the study of categories of sheaves: sheaves on a space, a locale, a site. This is the opportunity for developing the essential aspects of the theory of locales and introducing Grothendieck toposes. We relate this with the algebraic aspects of volume 2 by proving in this context the existence of a classifying topos for coherent theories. All these considerations lead naturally to the notion of an elementary topos. We study quite extensively the internal logic of toposes, including the law of excluded middle and the axiom of infinity. We conclude by showing how toposes are a natural context for defining sheaves.

Besides a technical development of the theory, many people appreciate historical notes explaining how the ideas appeared and grew. Let me tell you a story about that.

It was in July, I don’t remember the year. I was participating in a summer meeting on category theory at the Isles of Thorns, in Sussex. Somebody was actually giving a talk on the history of Eilenberg and Mac Lane’s collaboration in the forties, making clear what the exact contribution of the two authors was. At some point, somebody in the audience started to complain about the speaker giving credit to Eilenberg and Mac Lane for some basic aspect of their work which – he claimed – they borrowed from somebody else. A very sophisticated and animated discussion followed, which I was too ignorant to follow properly. The only things I can remember are the names of the two opponents: the speaker was Saunders Mac Lane and his opponent was Samuel Eilenberg. I was not born when they invented category theory. With my little story in mind, maybe you will forgive me for not having tried to give credit to anybody for the notions and results presented in this Handbook.

Let me conclude this introduction by thanking the various typists for their excellent job and my colleagues of the Louvain-la-Neuve category seminar for the fruitful discussions we had on various points of this Handbook. I want especially to acknowledge the numerous suggestions Enrico Vitale has made for improving the quality of my work.
Handbook of categorical algebra

Contents of the three volumes

Volume 1: Basic category theory
1. The language of categories
2. Limits
3. Adjoint functors
4. Generators and projectives
5. Categories of fractions
6. Flat functors and Cauchy completeness
7. Bicategories and distributors
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Volume 2: Categories and structures
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