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111 Affine differential geometry

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Geometry of Affine Immersions

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CONTENTS

Preface

Affine differential geometry has undergone a period of revival and rapid progress in the last ten years. Our purpose in writing this book is to provide a systematic introduction to the subject from a contemporary viewpoint, to cover most of the important classical results, and to present as much of the recent development as possible within the limitation of space. Rather than an encyclopedic collection of facts, we wanted to write an account of what we consider significant features of the subject including, wherever possible, the relationship to other areas of differential geometry. The Introduction will give the reader a brief history of the subject and the range of topics taken up in this book.

We should like to acknowledge our indebtedness to various institutions that made possible our cooperation with each other and with many other mathematicians working in affine differential geometry. First of all, to the Max-Planck Institut für Mathematik, where both authors did earlier work in the subject and later returned to do joint work, to Brown University, where the second author was visiting professor in 1988, to the Technische Universität Berlin, where we both were visitors in 1990 and where the first author worked in 1991 with a research grant from the Alexander von Humboldt Foundation, to the Mathematisches Forschungsinstitut Oberwolfach for the two conferences in affine differential geometry in 1986 and 1991, to the Katholieke Universiteit Leuven, where the first author had the pleasure of visiting for lectures and research, to Sichuan University, where the second author visited in 1992, and finally to Kobe University, where the first author was warmly received during the time we worked on the last stage of preparation for the current book – to all these institutions we express our sincere gratitude. With equally warm feelings we salute many international colleagues with whom we have cooperated in these several years: U. Pinkall, U. Simon, An-Min Li, L. Verstraelen, F. Dillen, L. Vrancken, B. Opozda, F. Podestà, A. Martínez, A. Magid, and many others. T. Cecil and M. Yoshida read the draft and offered helpful suggestions. C. Lee produced graphics with Mathematica on which the illustrations in the book are based. To all these colleagues and friends, we express our thanks for sharing their ideas.

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Introduction

Affine differential geometry – its history and current status

I. Before 1950

In 1908 Tzitzéica [Tz] showed that for a surface in Euclidean 3-space the ratio of the Gaussian curvature to the fourth power of the support function from the origin o is invariant under an affine transformation fixing o. He defined an S-surface to be any surface for which this ratio is constant. These S-surfaces turn out to be what are now called proper affine spheres with center at o. (See Theorem 5.11 and Remark 5.1 of Chapter II.) Blaschke, in collaboration with Pick, Radon, Berwald, and Thomsen among others, developed a systematic study of affine differential geometry in the period from 1916 to 1923, his work culminating in the treatise [Bl]. We wish to call attention to the two C.R. notes by E. Cartan [Car1,2] on affine surfaces in 1924 and to the work by Ślebodziński [Sl1,2] on locally symmetric affine surfaces. (See Note 8.) The main stream opened up by the Blaschke School was, however, followed by Kubota, Süss, Su Buging, Nakajima and others, whose work was mostly published in *Tôhoku Math. Journal* and the *Japanese Journal of Math.* in the late twenties and the thirties.

The main object in classical affine differential geometry is the study of properties of surfaces in 3-dimensional affine space that are invariant under the group of unimodular affine (or equiaffine) transformations. The ambient space $\mathbf{R}^3$ has a flat affine connection D and the usual determinant function Det regarded as a parallel volume element. A generalization of the theory to hypersurfaces followed almost immediately, but relatively little has been achieved in the study of affine submanifolds with codimension greater than 1. Burstin and Mayer [BM] studied surfaces M^2 in $\mathbf{R}^4$. See also Weise [Wei], later followed by Klingenberg [Kl1, 2].

Before we indicate how to study a hypersurface M^n in $\mathbf{R}^{n+1}$ from the equiaffine point of view, we recall the Euclidean hypersurface theory. Using the Euclidean metric in $\mathbf{R}^{n+1}$ we choose a field of unit normal vectors N to M^n. If X and Y are vector fields on M^n, we take the covariant derivative

$D_X Y$ in $\mathbf{R}^{n+1}$ and decompose it in the form

$$D_X Y = \nabla_X Y + h(X, Y)N,$$

where $\nabla_X Y$ is the tangential component and h the second fundamental form. It turns out that $\nabla_X Y$ is nothing but the covariant derivative relative to the Levi-Civita connection of the Riemannian metric induced on M^n. Now in the affine case, we simply take an arbitrary transversal vector field ξ and define ∇ and h in the same way as above. We can easily check that h is determined up to a conformal factor, and hence the rank of h depends only on the surface and not on the choice of ξ. It makes sense to say that M^n is a nondegenerate surface if h has rank n. Now for a nondegenerate hypersurface M^n, we can prove that there is a unique choice of ξ (up to sign) such that the induced volume element

$$\theta(X_1, \ldots, X_n) = \det [X_1 \ \ldots \ X_n \ \xi]$$

is parallel relative to the induced connection ∇ and, moreover, is identical with the volume element of the nondegenerate metric h. We prove this fundamental fact in Section 3 of Chapter II, and provide the *procedure for finding the affine normal* for practical computation. Such ξ is called the affine normal field, and the corresponding h the affine metric. The affine shape operator S is defined by $D_X \xi = -SX$. We get an equiaffine structure (∇, θ) on M^n; together with h and S, it will be called the Blaschke structure.

Affine spheres are those surfaces for which S is a scalar multiple of the identity, but unlike the Euclidean case they are by no means simple or easy to determine. Another important object is the cubic form $C = \nabla h$; it satisfies the property that $\mathrm{trace}_h(\nabla_X h) = 0$ for each tangent vector X, called apolarity. A classical theorem of Pick and Berwald states that a nondegenerate surface has vanishing cubic form if and only if it is a quadric. We give two proofs to this theorem in Section 4 of Chapter II and later provide generalizations in Sections 6 and 8 of Chapter IV. Blaschke also showed that a compact affine sphere is an ellipsoid (see Theorem 7.5 of Chapter III). An ellipsoid was also characterized as an ovaloid whose affine Gaussian curvature $K = \det S$ or affine mean curvature $H = \frac{1}{n}\mathrm{trace}\, S$ is constant. (See Section 9 of Chapter III.) Radon [R] gave a version of the fundamental theorem for surfaces, that is, the realization theorem of a 2-manifold with given h and C as a nondegenerate surface. We give a modern formulation for an arbitrary dimension in Theorem 8.2 of Chapter II. Thomsen studied minimal surfaces in Euclidean 3-space which are also minimal in the sense that $H = 0$. Affine ruled surfaces and affine minimal surfaces were studied from the local point of view.

II. Before 1980

Throughout the 1960s and 1970s a few expository papers and books appeared on the subject. In particular, we mention Schirokow and Schirokow

[Schr], 1962, as a very useful source of extensive information. Guggenheimer [Gu], 1963, has a chapter on affine differential geometry, in which he provides a classification list of equiaffine homogeneous surfaces. (This list has been completed in [NS2] by adding one more model, see Section 3 of Chapter III.) Spivak's volume IV [Sp] in 1975 contains a good account of affine surface theory. Schneider [Schn1,2] obtained interesting global results, and Simon [Si1–3] emphasized the viewpoint of so-called relative geometry that originates with Süss. See also Chern [Ch3], Flanders [Fl], Hsiung and Shahin [HS].

It was Calabi [Cal1–3] who made several major contributions dealing, for the first time, with global problems concerning noncompact affine hyperspheres. When a locally strictly convex hypersurface (that is, the affine metric is definite at each point) is a proper affine hypersphere (that is, the affine normals meet at one point, called the center), it is said to be elliptic or hyperbolic depending on whether the center lies on the concave side or the convex side of the hypersurface. Calabi's major results are the following.

(i) An improper affine hypersphere whose affine metric is complete is an elliptic paraboloid. (See Theorem 11.5 of Chapter III.)

(ii) A complete elliptic affine hypersphere is compact and hence an ellipsoid. (See Theorem 7.6 of Chapter III. Refer also to [Pog].)

(iii) His conjecture on hyperbolic affine hyperspheres was studied by Cheng and Yau [CY1] and Gigena [Gi] as well as by Sasaki [S1]. The result can be stated as follows: Every closed hyperbolic affine hypersphere is asymptotic to the boundary of a convex cone. Conversely, every nondegenerate cone V determines a hyperbolic affine hypersphere. We are not, however, going to give a proof of this result in our book. (See Note 7.)

We also mention some results by Chern and Terng [ChT] on affine minimal surfaces and affine Bäcklund theory as well as on the following result: A surface in Euclidean space $\mathbf{R}^3$ that is isometric to part of the elliptic paraboloid has zero affine mean curvature. This peculiar mixture of Euclidean and affine properties seems so far to be an isolated result (except that [NS1] gave a similar result regarding a timelike surface in the Minkowski space with $ds^2 = dx^2 - dy^2 + dz^2$ isometric to $z = x^2 - y^2$). We should also mention the important progress being made by Calabi [Cal4] on the so-called affine Bernstein problem, formulated in [Ch3] as follows. If the graph $z = f(x, y)$ defined on the entire plane is a nondegenerate surface with zero affine mean curvature, is it affinely congruent to the graph of $z = x^2 + y^2$? See Section 11 of Chapter III. Most of the important aspects of the classical theory are treated in Chapters II and III, together with some of the results based on the new approach to the subject described in the next section.

III. After 1980

In the *Complete Works* of Blaschke, two survey articles on his work in affine differential geometry as well as later developments appeared; see Burau and Simon [BS] and Simon [Si4].

In the lecture entitled 'What is affine differential geometry?' at the Münster Conference in 1982, Nomizu formulated the starting point of affine differential geometry from the structural point of view. After discussing $SL(n, \mathbf{R})$ as an example of a nondegenerate affine hypersurface and observing that the induced connection on $SL(2, \mathbf{R})$ is locally symmetric, he posed a problem: Determine the nondegenerate hypersurfaces whose induced connections are locally symmetric. Verheyen and Verstraelen [VeVe] solved the problem by showing that such hypersurfaces M^n, $n \geq 3$, must have either $S = 0$ (hence the graph of a function) or $C = 0$ (hence a quadric). This result has now been extended, as stated in Section 4 of Chapter IV. Such a generalization was possible only after the notion of affine immersion, introduced in [NP2], gave a more general viewpoint to the subject. We treat this notion in Sections 1 and 2 in Chapter II. An affine immersion is totally geodesic if h is identically 0; it is a graph immersion if S is identically 0. An isometric immersion may be regarded as an affine immersion. It is possible to allow the affine fundamental form h to be degenerate; in fact, many classical results can be extended to the case where the rank of h is at least equal to 2. (For example, see the extensions of the Pick–Berwald theorem in Section 6 of Chapter IV.) We may study the extreme situation of an affine immersion $f : \mathbf{R}^n \to \mathbf{R}^{n+1}$ as an analogue to the problem of determining all isometric immersions between Euclidean spaces E^n and E^{n+1} or between Minkowski–Lorentzian spaces L^n and L^{n+1}. Actually, the affine method can unify the metric cases, as shown in Section 2 of Chapter IV. Another bridge from the affine theory to the metric geometry is the Cartan–Norden theorem in [NP2], which is contained in Section 3 of Chapter IV. Opozda [O7] has studied the structures of Blaschke surfaces in $\mathbf{R}^3$ with locally symmetric induced connections and further the problem of realizing locally symmetric (or more generally, projectively flat) connections on 2-manifolds as induced connections on nondegenerate surfaces; these results are sketched in Note 8.

From the structural point of view, we mention two problems. One is the formulation for a generalization of Radon's theorem. On a given M^n assume that there is a nondegenerate metric h and a cubic form C. Under what conditions can we realize h and C as the affine metric and the cubic form for a nondegenerate immersion of M^n into $\mathbf{R}^{n+1}$? Such a pair (h, C) corresponds to a pair (h, ∇) satisfying Codazzi's equation (that ∇h is symmetric in all three variables). In addition to apolarity, the condition is that the conjugate connection $\bar{\nabla}$ of ∇ relative to h is projectively flat. This was established in [DNV]. We also note that the notion of conjugate connection has proved to be independently important due to its applications to mathematical statistics (see Amari [Am1]). We cannot go into this application, but quote a few references by Amari [Am2] and Kurose [Ku2–5] motivated by such applications.

The other problem is the rigidity theorem of Cohn-Vossen type given in [NO1]. The classical Cohn-Vossen theorem says that two compact convex surfaces (ovaloids) in Euclidean space are congruent if they are isometric. Since the affine theory deals with the induced connection instead of the induced metric, it is natural to anticipate a result that says that two ovaloids are affinely congruent if their induced connections are isomorphic. We prove this theorem, under the assumption that the affine Gauss–Kronecker curvature K does not vanish, by a method similar to the Herglotz proof of Cohn-Vossen's theorem (see Section 5 of Chapter IV). We note that Simon [Si7] has a different approach that does not require the condition on K.

Going back to the classical Blaschke immersions, we mention a few new results. An-Min Li [L5, 8] has proved that a locally strictly convex hypersphere M^n with complete affine metric is closed in $\mathbf{R}^{n+1}$, thereby strengthening [CY1] in the proof of the Calabi conjecture. He later proved that the affine metric of a closed, locally strictly convex hypersurface is complete if the affine principal curvatures are bounded [L7]. He also has contributed to the affine Bernstein problem by characterizing an elliptic paraboloid as a locally strictly convex, affine-complete, affine-minimal surface whose affine normals omit five or more directions in general position [L4]. Martínez and Milán have characterized an elliptic paraboloid as a locally strictly convex, affine-complete, affine-minimal surface with K bounded from below by a constant [MMi2].

The affine spheres with affine metrics of constant curvature have been classified (among them, ruled affine spheres, proper and improper), see Section 5, Chaper III, and the references therein for these and related results. As an extension of Theorem 5.1, Magid and Ryan [MR2] proved: If an affine hypersphere M^3 in $\mathbf{R}^4$ has nonzero Pick invariant and its affine metric has constant sectional curvature α, then α must be 0 and M^3 is affinely congruent to an open subset of one of the following models:

$$\text{(i) } x_1x_2x_3x_4 = 1; \quad \text{(ii) } (x_1^2 + x_2^2)(x_3^2 - x_4^2) = 1; \quad \text{(iii) } (x_1^2 + x_2^2)(x_3^2 + x_4^2) = 1.$$

We mention that Nomizu and Vrancken [NV] recently introduced the notion of nondegeneracy for surfaces M^2 in $\mathbf{R}^4$. Under this assumption, it is possible to induce an equiaffine connection on M^2 by choosing a suitable transversal bundle. There are good indications that this is the right choice of connection on nondegenerate surfaces with codimension 2 (see Note 2 and references therein). A good amount of parallel ideas appears between this affine theory and the projective theory in Sasaki [S7].

For other recent work, see many papers by Dillen, Verstraelen, Vrancken and their joint work, [Di1–4], [DVe], [DVr1–5], [DVV], [VeVr], and [Wan1, 2], [NR], [Ce].

Another application of the formalism of affine immersion is to the study of projective differential geometry. Three different approaches can be mentioned. The first is the study of projective immersion – an analogue of affine immersion. We study maps between two manifolds each with a projective structure; for example, automorphisms of (M^n, P) into itself or a hypersurface immersion of (M^n, P) into $(\tilde{M}^{n+1}, \tilde{P})$. They were studied in [NP3] and [NP4]. We deal with some of these results in Section 7 of Chapter III. The second is the study of nondegenerate immersions of an n-manifold M^n into the real projective space $\mathbf{P}^{n+1}$. By using an atlas of flat affine connections with parallel volume element (D,ω) on $\mathbf{P}^{n+1}$, we consider a neighborhood of each point of M^n as a Blaschke hypersurface. We look for "projective invariants" through this process, for example, the conformal class of h, the vanishing of the Pick invariant J, the projective metric when J is nonvanishing, etc. (See Section 8 of Chapter IV.) This makes it possible to define projective invariants that usually require the method of moving frames (see [S3, 4]). The third is the method of centro-affine immersions of codimension 2, which is applied to hypersurface theory in $\mathbf{P}^{n+1}$ through the canonical map $\mathbf{R}^{n+2} - \{0\} \to \mathbf{P}^{n+1}$ (see Note 9 and [NS5]). The theory of codimension-2 centro-affine immersions was developed also by Walter [Wal] without regard to projective hypersurface theory. The second and third methods are combined in order to classify projectively homogeneous surfaces in $\mathbf{P}^3$ (see Note 11 and [NS5]).

Finally, we mention that affine differential geometry has been extended to the complex case. Specifically, let $f : M^n \to \mathbf{C}^{n+1}$ be a holomorphic immersion of a complex manifold M^n of complex dimension n into complex affine space $\mathbf{C}^{n+1}$. K. Abe [Ab] and Dillen, Verstraelen and Vrancken [DVV] considered an analogue of the real case by choosing a holomorphic transversal vector field and getting an induced holomorphic connection on M^n. See also [Di1,2], [DVr1,2], [DVe]. On the other hand, Nomizu, Pinkall and Podestà [NPP] chose an anti-holomorphic transversal vector field and found on M^n an affine-Kähler connection, that is, a connection ∇ compatible with the complex structure J on M^n whose curvature tensor satisfies $R(JX,JY) = R(X,Y)$ for all tangent vectors X, Y. (We note that the Levi-Civita connection of a Kähler metric is affine-Kähler. If an affine-Kähler connection is also holomorphic, then it is flat.) For this topic, see also [NPo1,2], [Po1].

Opozda has dealt with a general theory that includes the holomorphic and anti-holomorphic choices of transversal vector fields. See [O5,6], and [O8]. We treat complex affine geometry in Section 9 of Chapter IV.