I

Affine geometry and affine connections

This chapter summarizes what will be needed in the main body of the book. Most of the material here can be recalled from the standard references on geometry and on differential geometry; basically, we shall suppose that the reader is familiar with the terminology in manifold theory. Some of the topics, however, are already part of an introduction to affine differential geometry: affine curves in Section 1, the notions of equiaffine connection and centro-affine surface in Section 3, and the notions of conjugate connection and cubic form in Section 4. In Section 5 we briefly discuss vector bundles and interpret the conjugate connection from the point of view of the cotangent bundle.

1. Plane curves

In this section, we provide a preview of affine differential geometry through elementary discussions of curves in the affine plane. The affine plane $\mathbf{R}^2$ is referred to in terms of a coordinate system $\{x, y\}$. By an allowable coordinate change, we mean $(x, y) \mapsto (\bar{x}, \bar{y})$ such that

$$\bar{x} = ax + by + p, \qquad \bar{y} = cx + dy + q, \tag{1.1}$$

where

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = 1. \tag{1.2}$$

Equiaffine geometry is concerned with geometric properties expressed by conditions invariant under an allowable change of coordinates. Or we may consider (1.1) as an allowable transformation of the plane $\mathbf{R}^2$, called an *equiaffine transformation*; then equiaffine geometry is the study of properties invariant under equiaffine transformations. Geometrically, we may assume that the notion of parallelogram of unit area is given. A coordinate system is allowable if the parallelogram determined by $(1,0)$ and $(0,1)$ has unit area.

This being said, we consider a smooth curve

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix},$$

where $x(t)$, $y(t)$ are smooth functions of t defined over a certain interval J. We denote by $\mathbf{a}_1(t)$ the tangent vector $\dot{\mathbf{x}} = d\mathbf{x}/dt$ and look for a second vector $\mathbf{a}_2(t)$ along the curve $\mathbf{x}(t)$ such that the vectors $\mathbf{a}_1(t)$ and $\mathbf{a}_2(t)$ span unit area, that is, $\det[\mathbf{a}_1(t)\ \mathbf{a}_2(t)] = |\mathbf{a}_1(t)\ \mathbf{a}_2(t)| = 1$.

We can actually try to do better. We look for a new parameter σ for the curve $\mathbf{x}(t)$ such that

$$[\mathbf{x}'\ \mathbf{x}''] \in SL(2, \mathbf{R}),$$

where $'$ denotes the derivative with respect to σ, and $SL(2, \mathbf{R})$ is the group of all 2×2 real matrices with determinant 1. We have

$$|\dot{\mathbf{x}}\ \ddot{\mathbf{x}}| = (\dot{\sigma})^3\, |\mathbf{x}'\ \mathbf{x}''| = \left(\frac{d\sigma}{dt}\right)^3.$$

Thus

$$\frac{d\sigma}{dt} = |\dot{\mathbf{x}}\ \ddot{\mathbf{x}}|^{\frac{1}{3}}, \tag{1.3}$$

and hence

$$\sigma(t) = \int_{t_0}^{t} |\dot{\mathbf{x}}\ \ddot{\mathbf{x}}|^{\frac{1}{3}}\, dt. \tag{1.4}$$

In order for (1.4) to define a reparametrization $t \mapsto \sigma$, we shall assume that the given curve $\mathbf{x}(t)$ satisfies the condition

$$|\dot{\mathbf{x}}\ \ddot{\mathbf{x}}| \neq 0 \quad \text{for any } t. \tag{1.5}$$

The reader may easily verify that this condition is independent of parametrization. We say that the curve is *nondegenerate*. To see the geometric meaning of the condition we suppose $|\dot{\mathbf{x}}\ \ddot{\mathbf{x}}| > 0$ for all t. Then for small h we have

$$\mathbf{x}(t+h) - \mathbf{x}(t) = h\dot{\mathbf{x}}(t) + \frac{1}{2}h^2\ddot{\mathbf{x}}(\tau),$$

where τ is between t and $t+h$. Hence we get

$$|\dot{\mathbf{x}}(t)\quad \mathbf{x}(t+h) - \mathbf{x}(t)| = \frac{1}{2}h^2|\dot{\mathbf{x}}(t)\ \ \ddot{\mathbf{x}}(\tau)| > 0.$$

This means that the secant from $\mathbf{x}(t)$ to $\mathbf{x}(t+h)$ lies on one side of the tangent, that is, all points of the curve near $\mathbf{x}(t)$ lie on one side of the tangent at $\mathbf{x}(t)$. Thus our condition means that the curve has no inflection points.

We have shown that a nondegenerate curve $\mathbf{x}(t)$ admits a parameter σ such that

$$A(\sigma) = [\mathbf{x}' \ \mathbf{x}''] \in SL(2, \mathbf{R}).$$

From (1.3) it is clear that such a parameter, called an *affine arclength parameter*, is unique up to a constant. We define the *affine curvature* of the curve by

$$\kappa(\sigma) = |\mathbf{x}'' \ \mathbf{x}'''|. \tag{1.6}$$

Differentiating $|\mathbf{x}' \ \mathbf{x}''| = 1$, we obtain $|\mathbf{x}' \ \mathbf{x}'''| = 0$ and hence

$$\mathbf{x}''' = -\kappa \mathbf{x}'. \tag{1.7}$$

Note that the affine arclength parameter and affine curvature are invariant under an equiaffine transformation of the plane: if $\mathbf{x}(\sigma)$ is a curve with affine arclength parameter σ, then for any equiaffine transformation f the curve $\mathbf{y}(\sigma) = f(\mathbf{x}(\sigma))$ is a nondegenerate curve with σ as its affine arclength parameter; moreover, the affine curvature of $\mathbf{y}(\sigma)$ is equal to the affine curvature of $\mathbf{x}(\sigma)$ for each value of σ.

To give a Lie group-theoretic interpretation, we define the matrix $C(\sigma) = A(\sigma)^{-1} dA/d\sigma$. By virtue of (1.7), we get

$$\frac{dA}{d\sigma} = A(\sigma) \begin{bmatrix} 0 & -\kappa \\ 1 & 0 \end{bmatrix},$$

that is,

$$C(\sigma) = \begin{bmatrix} 0 & -\kappa \\ 1 & 0 \end{bmatrix}. \tag{1.8}$$

Theorem 1.1. *Given a smooth function $\kappa = \kappa(\sigma)$ on an interval J, there exists a plane curve with σ as affine arclength parameter and κ as affine curvature. Such a curve is unique up to equiaffine transformation of the plane.*

Proof. We consider the matrix-valued function $C(\sigma)$ as in (1.8) with the given $\kappa(\sigma)$. From the existence theorem for ordinary differential equations, we have a unique solution of the equation

$$\frac{dA}{d\sigma} = A(\sigma)C(\sigma), \tag{1.9}$$

with the initial condition $A(0) = I_2$ (identity matrix of degree 2). Using the fact that $C(\sigma)$ has trace 0 we can verify that $A(\sigma)$ is a curve in the group $SL(2, \mathbf{R})$. By writing $A(\sigma) = [\mathbf{a}_1(\sigma) \ \mathbf{a}_2(\sigma)]$ and integrating $\mathbf{a}_1(\sigma)$ we obtain a desired curve $\mathbf{x}(\sigma)$. We leave the proof of uniqueness to the reader.

Remark 1.1. Theorem 1.1 is the affine analogue of the fundamental theorem for curves in the Euclidean plane, where the uniqueness is up to Euclidean isometry. We call $\kappa = \kappa(\sigma)$ a *natural equation* for the curve.

Let us consider the special case where κ is a constant function.

First, assume $\kappa = 0$. Thus $\mathbf{x}''' = 0$, from which we obtain

$$\mathbf{x} = \frac{1}{2}\sigma^2\mathbf{b} + \sigma\mathbf{a} + \mathbf{c},$$

where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are constant vectors such that $|\mathbf{a}\ \mathbf{b}| = 1$. By taking the coordinate system $\{x, y\}$ such that

$$\mathbf{a} = (1,0),\ \mathbf{b} = (0,1),\ \mathbf{c} = (0,0)$$

we see that the curve is a *parabola* $y = \frac{1}{2}x^2$.

Second, assume that $\kappa > 0$. We have then

$$(\mathbf{x}')'' + (\sqrt{\kappa})^2\mathbf{x}' = 0.$$

We obtain

$$\mathbf{x}' = \cos(\sqrt{\kappa}\sigma)\mathbf{a} + \sin(\sqrt{\kappa}\sigma)\mathbf{b},$$

where $\mathbf{x}'(0) = \mathbf{a}$ and $\mathbf{x}''(0) = \sqrt{\kappa}\,\mathbf{b}$ so that $|\mathbf{a}\ \mathbf{b}| = 1/\sqrt{\kappa}$. Choosing a coordinate system such that $\mathbf{a} = (1,0)$ and $\sqrt{\kappa}\,\mathbf{b} = (0,1)$, we have

$$\mathbf{x} = \left(\frac{\sin(\sqrt{\kappa}\sigma)}{\sqrt{\kappa}}, -\frac{\cos(\sqrt{\kappa}\sigma)}{\kappa}\right).$$

The curve is an ellipse

(1.10) $$\kappa x^2 + \kappa^2 y^2 = 1, \quad \kappa > 0.$$

Finally, in the case where $\kappa < 0$, we obtain the curve in the form

$$\mathbf{x} = \left(\frac{\sinh(\sqrt{-\kappa}\sigma)}{\sqrt{-\kappa}}, \frac{\cosh(\sqrt{-\kappa}\sigma)}{-\kappa}\right),$$

that is, a hyperbola

(1.11) $$\kappa x^2 + \kappa^2 y^2 = 1, \quad \kappa < 0.$$

To sum up, we have

Theorem 1.2. *If a nondegenerate curve has constant curvature κ, it is a quadratic curve.*

We also state

Example 1.1. Relative to an allowable affine coordinate system, consider the curves

$$\mathbf{x} = (a\cos t,\ b\sin t)$$

and

$$\mathbf{x} = (a\cosh t,\ b\sinh t).$$

Their affine curvatures are $(ab)^{-\frac{2}{3}}$ and $-(ab)^{-\frac{2}{3}}$, respectively.

We say that a plane curve is *homogeneous* if it is the orbit of a certain point under a 1-parameter group of equiaffine transformations. A homogeneous nondegenerate curve has constant affine curvature because from one point p to another point q of the curve there is an equiaffine transformation of the plane that takes p to q and leaves the curve invariant. Since affine curvature is an equiaffine property, it remains constant on the curve. Conversely, a nondegenerate curve of constant affine curvature is homogeneous. For example, an *ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the orbit of the point $(a, 0)$ under the 1-parameter group of equiaffine transformations

$$A(t) = \begin{bmatrix} \cos t & -\lambda \sin t \\ \frac{1}{\lambda}\sin t & \cos t \end{bmatrix}, \tag{1.12}$$

where $\lambda = a/b$. For a *hyperbola*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

we take

$$A(t) = \begin{bmatrix} \cosh t & \lambda \sinh t \\ \frac{1}{\lambda}\sinh t & \cosh t \end{bmatrix}. \tag{1.13}$$

For a parabola

$$y = \frac{1}{2}x^2,$$

we can take the 1-parameter group of equiaffine transformations

$$\tilde{A}(t) = \begin{bmatrix} 1 & 0 & t \\ t & 1 & \frac{1}{2}t^2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{acting on} \quad \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{1.14}$$

and consider the orbit of the origin $(0, 0)$.

We can thus state

Theorem 1.3. *A nondegenerate curve has constant affine curvature if and only if it is the orbit of a point under a certain 1-parameter group of equiaffine transformations.*

We shall refer to [Bl] for the following result.

Theorem 1.4. *The 1-parameter groups of equiaffine transformations of the plane can be classified as follows, up to affine equivalence.*

(A) $$\begin{bmatrix} \cos t & -\sin t & 0 \\ \sin t & \cos t & 0 \\ 0 & 0 & 1 \end{bmatrix};\quad \textit{same as } (1.12) \textit{ with } \lambda = 1.$$

(B) $$\begin{bmatrix} \cosh t & \sinh t & 0 \\ \sinh t & \cosh t & 0 \\ 0 & 0 & 1 \end{bmatrix};\quad \textit{same as } (1.13) \textit{ with } \lambda = 1.$$

(C) $$\begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(D) $$\begin{bmatrix} 1 & 0 & t \\ t & 1 & \frac{1}{2}t^2 \\ 0 & 0 & 1 \end{bmatrix};\quad \textit{same as } (1.14).$$

(E) $$\begin{bmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We shall derive a few geometric results by using the equiaffine properties of these transformations.

Example 1.2. Let us consider two parabolas

$$y = \frac{1}{2}x^2 \quad \text{and} \quad y = \frac{1}{2}x^2 + c^2.$$

The transformation group (D) maps each parabola onto itself transitively. Take a point P on the second parabola and let the tangent at P meet the first parabola at M and N. Then the region G bounded by the first parabola and the line MN has constant area independent of the choice of P. To see this, let P' be any other point and construct the points M' and N'. The transformation (D) that takes P to P' will map M and N upon M' and N' and the region G onto the other region G'. Since (D) is equiaffine, we see that G and G' have the same area. See Figure 1(a).

Example 1.3. Consider two ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = k,\ k > 1.$$

For a point P on the first ellipse, let the tangent at P meet the second ellipse at M and N. Then the region G bounded by the second ellipse and the line

MN outside the first ellipse has constant area independent of P. We use the group (1.12). See Figure 1(b).

Example 1.4. A similar result is valid for two hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = k,\ k > 1.$$

Example 1.5. For the hyperbola $xy = 1$, $x > 0$, let the tangent at a point P meet the x-axis and the y-axis at M and N. The triangle $\triangle OMN$, O being the origin, has constant area independent of P. Here we use the group $\begin{bmatrix} a & 0 \\ 0 & 1/a \end{bmatrix}$, $a > 0$, that maps each of the hyperbola $xy = 1$, the half-line $y = 0$, $x > 0$, and the half-line $x = 0$, $y > 0$ onto itself transitively.

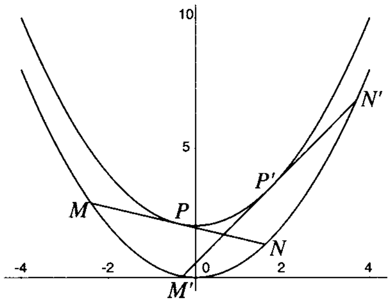

Figure 1(a)

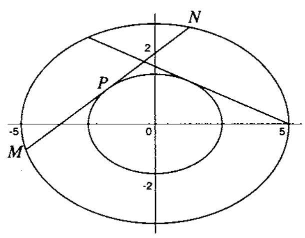

Figure 1(b)

2. Affine space

In this section we first give a rigorous definition of affine space of an arbitrary dimension by using Weyl's axioms. We then proceed to define related concepts and results.

Definition 2.1. Let V be a real n-dimensional vector space. A non-empty set Ω is said to be an *affine space* associated to V if there is a mapping

$$\Omega \times \Omega \to V$$

denoted by

$$(p, q) \in \Omega \times \Omega \mapsto \overrightarrow{pq} \in V$$

satisfying the following axioms:

(2.1) for any $p, q, r \in \Omega$, we have $\overrightarrow{pr} = \overrightarrow{pq} + \overrightarrow{qr}$;

(2.2) for any $p \in \Omega$ and for any $x \in V$ there is one and only one $q \in \Omega$ such that $x = \overrightarrow{pq}$.

It is also convenient to write $q = p + x$ instead of $x = \overrightarrow{pq}$, particularly when we are given the point p and the vector x and wish to denote a point q as in (2.2). The *dimension* of Ω is defined as that of V. We also say that V is the *associated vector space* for the affine space Ω.

Example 2.1. Let V be a real vector space of dimension n. Consider V as a set and, for $(p,q) \in V \times V$, define $x = \overrightarrow{pq}$ to be the vector $q - p \in V$. In this way, V becomes an n-dimensional affine space.

Example 2.2. In particular, let V be the standard real n-dimensional vector space $\mathbf{R}^n$. Then regard it as an affine space in the manner of Example 2.1. We call it the *standard n-dimensional affine space.*

The simple axioms above, due to H. Weyl, are sufficient to derive all the properties of an affine space. We shall outline how they can be developed. First, observe that we have $\overrightarrow{pp} = 0$ and $\overrightarrow{qp} = -\overrightarrow{pq}$ for any $p, q \in \Omega$.

Definition 2.2. An affine coordinate system with origin $o \in \Omega$ can be defined as follows. Let $\{e_1, \ldots, e_n\}$ be a basis of V. For any point $p \in \Omega$, we write

$$\overrightarrow{op} = \sum_{i=1}^{n} x^i(p) e_i, \tag{2.3}$$

where $(x^1(p), \ldots, x^n(p))$ is a uniquely determined n-tuple of real numbers, called the *coordinates* of p. The set of functions $\{x^1, \ldots, x^n\}$ is called an *affine coordinate system.*

If we have two affine coordinate systems $\{x^1, \ldots, x^n\}$ and $\{y^1, \ldots, y^n\}$, then they are related by

$$y^i = \sum a^i_j x^j + c^i, \ 1 \leq i \leq n, \tag{2.4}$$

where $A = [a^i_j]$ is a nonsingular $n \times n$ matrix and $c = [c^i]$ is a vector. This relation may be expressed by the equation $y = Ax + c$, or in its expanded matrix form,

$$\begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

We now define the notion of affine transformation. Let $f : \Omega \to \Omega$ be a one-to-one mapping of Ω onto itself. For each $p \in \Omega$ we define a mapping $F_p : V \to V$ as follows. For each $x \in V$, let $r \in \Omega$ be a uniquely determined point in Ω such that $\overrightarrow{pr} = x$. Then we set $F_p(x) = \overrightarrow{f(p)f(r)}$.

Definition 2.3. We say that f is an *affine transformation* if, for a certain $p \in \Omega$, the map F_p is a linear transformation of V onto itself (and it is nonsingular, since it is one-to-one together with f). In this case, it follows that for any point $q \in \Omega$ the map F_q coincides with F_p. We can therefore call this map the *associated linear transformation* and denote it simply by F.

Let $\{x^1, \ldots, x^n\}$ be an affine coordinate system with origin o and based on a basis $\{e_1, \ldots, e_n\}$. Let $\{y^1, \ldots, y^n\}$ be the affine coordinate system with origin $f(o)$ and based on the basis $\{F(e_1), \ldots, F(e_n)\}$. Then we have

$y_i(f(p)) = x_i(p)$ for $p \in \Omega$. We can write the relationship between the coordinate systems $\{x^1, \dots, x^n\}$ and $\{y^1, \dots, y^n\}$ in the form

$$x^i = \sum b^i_j y^j + d^i,\ 1 \le i \le n. \tag{2.5}$$

Therefore, relative to the one coordinate system $\{x^1, \dots, x^n\}$, the coordinates $\bar{x}^i = x^i(f(p))$ of the image $f(p)$ can be expressed in terms of the coordinates of p in the form

$$\bar{x}^i = x^i(f(p)) = \sum b^i_j y^j(f(p)) + d^i = \sum b^i_j x^j(p) + d^i,\ 1 \le i \le n. \tag{2.6}$$

Again, this equation can be put in the form $\bar{x} = Bx + d$ or in its expanded matrix form,

$$\begin{bmatrix} \bar{x} \\ 1 \end{bmatrix} = \begin{bmatrix} B & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}.$$

We define the notion of affine subspace.

Definition 2.4. A nonempty subset Ω' of Ω is called an *affine subspace* if, for a certain point $p \in \Omega'$, the set of vectors $\{\overrightarrow{pq} : q \in \Omega'\}$ of Ω forms a vector subspace W of V. In this case, p in the condition can be replaced by any other point of Ω' with the same vector subspace W resulting. It follows that Ω' is an affine space associated to the vector space W. The dimension of an affine subspace is, by definition, the dimension of the associated vector space.

Example 2.3. Given two points $p, q \in \Omega$, the line pq is a 1-dimensional affine subspace $\{r \in \Omega : \overrightarrow{pr} = t\overrightarrow{pq} \text{ for } t \in \mathbf{R}\}$.

We now consider the standard real affine space $\Omega = \mathbf{R}^n$ as an n-dimensional differentiable manifold. For each point $p \in \mathbf{R}^n$ we may identify the tangent space $T_p(\Omega) = T_p(\mathbf{R}^n)$ with the vector space $V = \mathbf{R}^n$. This means that we consider each $x \in V$ as a geometric vector placed at p, that is, $\overrightarrow{pq}$ interpreted as the pair of initial point p and end point q. Furthermore, we may consider $x \in V$ as a vector field that assigns to each $p \in \Omega$ a tangent vector $\overrightarrow{pq}$ determined by x. Geometrically, all these vectors determined by x are parallel. From the construction of an affine coordinate system $\{x^1, \dots, x^n\}$ it follows that $\partial/\partial x^i$ as a vector field corresponds to e_i, where $\{e_1, \dots, e_n\}$ is a basis of V on which the affine coordinate system is based.

Let us now consider an arbitrary vector field Y on $\Omega = \mathbf{R}^n$ and let $x_t, a < t < b$, be an arbitrary smooth curve in Ω. We define the *covariant derivative* $D_t Y$ of Y along the curve x_t by

$$D_t Y = \lim_{h \to 0} \frac{1}{h}(\Pi Y_{x_{t+h}} - Y_{x_t}),$$

where Π denotes the identification map $T_{x_{t+h}}(\mathbf{R}^n) \to T_{x_t}(\mathbf{R}^n)$ through V in the manner explained above. If we take any affine coordinate system $\{x^1, \dots, x^n\}$ and write

$$Y = \sum_{i=1}^{n} Y^i \frac{\partial}{\partial x^i} \quad \text{and} \quad x_t = (x^1(t), \dots, x^n(t)),$$

then

$$D_t Y = \sum_{i=1}^{n} \frac{dY^i(x_t)}{dt} \frac{\partial}{\partial x^i} = \sum_{i,j=1}^{n} \frac{\partial Y^i}{\partial x^j} \frac{dx^j}{dt} \frac{\partial}{\partial x^i}.$$

Thus $D_t Y$ is a generalization of the directional derivative of functions to vector fields. If X is a tangent vector at a point x_0, then $D_X Y$ is defined by $D_X Y = (D_t Y)_t$, where x_t is a curve with initial point x_0 and initial tangent vector X. From this definition, it is clear that covariant differentiation D has the following properties:

(1) $D_{X_1+X_2} Y = D_{X_1} Y + D_{X_2} Y$;
(2) $D_{\phi X} Y = \phi D_X Y$;
(3) $D_X(Y_1 + Y_2) = D_X Y_1 + D_X Y_2$;
(4) $D_X(\phi Y) = (X\phi)Y + \phi D_X Y$;

where ϕ is a smooth function and X, Y, X_1, X_2, Y_1, and Y_2 are vector fields on Ω.

We now consider the notion of parallel volume element in $\Omega = \mathbf{R}^n$. First we fix a *volume element* ω in the vector space $V = \mathbf{R}^n$. This is nothing but a nonzero alternating n-form; once an orientation of V is fixed, it is determined up to a positive constant factor, that is, for any oriented basis $\{e_1, \ldots, e_n\}$, the value $\omega(e_1, \ldots, e_n)$ can be assigned to be an arbitrary positive number c, which determines ω uniquely.

A volume element ω in V determines a volume element on the manifold Ω, that is, a nonvanishing differential n-form, denoted by the same letter ω, such that

$$\omega\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = c,$$

where the vector fields $\partial/\partial x^1, \ldots, \partial/\partial x^n$ correspond to $e_1, \ldots, e_n$ as explained above. It is obvious that $\omega(X_1, \ldots, X_n) = \omega(Y_1, \ldots, Y_n)$ if each $Y_i \in T_y(\mathbf{R}^{n+1})$ is parallel to $X_i \in T_x(\mathbf{R}^{n+1})$. Hence ω is said to be *parallel*.

Once a parallel volume element is fixed in the affine space Ω, an affine transformation f is said to be *equiaffine* (or *unimodular*) if it preserves the volume element, that is, the associated linear transformation F preserves the corresponding volume element in V. It is easy to verify that this is the case if and only if f is expressed by (2.6), where the matrix $[b^i_j]$ has determinant 1. The set of all equiaffine transformations forms a subgroup of the group of all affine transformations.

The geometry of submanifolds of an affine space is called *affine differential geometry*. We study the properties that are invariant under the group of affine transformations, just as Euclidean differential geometry is the geometry of submanifolds of a Euclidean space in which we study the properties invariant under Euclidean isometries. In affine differential geometry, particularly important is the study of properties invariant under equiaffine transformations. We have already seen an elementary theory of curves from this point of view; for example, we have characterized the quadrics in $\mathbf{R}^2$ by constancy