Random Graphs

The book is devoted to the study of classical combinatorial structures, such as random graphs, permutations, and systems of random linear equations in finite fields. The author shows how the application of the generalized scheme of allocation in the study of random graphs and permutations reduces the combinatorial problems to classical problems of probability theory on the summation of independent random variables. He concentrates on recent research by Russian mathematicians, including a discussion of equations containing an unknown permutation. This is the first English-language presentation of techniques for analyzing systems of random linear equations in finite fields.

These results will interest specialists in combinatorics and probability theory and will also be useful in applied areas of probabilistic combinatorics, such as communication theory, cryptology, and mathematical genetics.

V. F. Kolchin is a leading researcher at the Steklov Institute and a professor at the Moscow Institute of Electronics and Mathematics (MIEM). He has written four books and many papers in the area of probabilistic combinatorics. His papers have been published mainly in the Russian journals *Theory of Probability and Its Applications*, *Mathematical Notes*, and *Discrete Mathematics*, and in the international journal *Random Structures and Algorithms*.

ENCYCLOPEDIA OF MATHEMATICS AND ITS APPLICATIONS

EDITED BY G.-C. ROTA

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CAMBRIDGE
UNIVERSITY PRESS

CAMBRIDGE UNIVERSITY PRESS
Cambridge, New York, Melbourne, Madrid, Cape Town, Singapore, São Paulo

Cambridge University Press
The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org
Information on this title: www.cambridge.org/9780521440813

First published 1999

A catalogue record for this publication is available from the British Library

Library of Congress Cataloguing in Publication data
Kolchin, V. F. (Valentin Fedorovich)
Random graphs / V. F. Kolchin
p. cm. – (Encyclopedia of mathematics and its applications; v. 53)
Includes bibliographical references and index.
ISBN 0 521 44081 5 hardback
1. Random graphs. I. Title. II. Series.
QA166.17.K65 1999 98-24390
511′.5 – dc20 CIP

ISBN 978-0-521-44081-3 hardback

Transferred to digital printing 2007

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PREFACE

Combinatorics played an important role in the development of probability theory and the two have continued to be closely related. Now probability theory, by offering new approaches to problems of discrete mathematics, is beginning to repay its debt to combinatorics. Among these new approaches, the methods of asymptotic analysis, which have been well developed in probability theory, can be used to solve certain complicated combinatorial problems.

If the uniform distribution is defined on the set of combinatorial structures in question, then the numerical characteristics of the structures can be regarded as random variables and analyzed by probabilistic methods. By using the probabilistic approach, we restrict our attention to "typical" structures that constitute the bulk of the set, excluding the small fraction with exceptional properties.

The probabilistic approach that is now widely used in combinatorics was first formulated by V. L. Goncharov, who applied it to S_n, the set of all permutations of degree n, and to the runs in random (0,1)-sequences. S. N. Bernstein, N. V. Smirnov, and V. E. Stepanov were among those who developed probabilistic combinatorics in Russia, building on the famous Russian school of probability founded by A. A. Markov, P. L. Lyapunov, A. Ya. Khinchin, and A. N. Kolmogorov.

This book is based on results obtained primarily by Russian mathematicians and presents results on random graphs, systems of random linear equations in GF(2), random permutations, and some simple equations involving permutations.

Selecting material for the book was a difficult job. Of course, this book is not a complete treatment of the topics mentioned. Some results (and their proofs) did not seem ready for inclusion in a book, and there may be relevant results that have escaped the author's attention.

There is a large body of literature on random graphs, and it is not possible to review it here. Among the probabilistic tools that have been used to analyze random structures are the method of moments, Poisson and Gaussian approximations, generating functions using the saddle-point method, Tauberian-type theorems, analysis of singularities, and martingale theory. In the past two decades, a method called the generalized scheme of allocation has been widely used in probabilistic combinatorics. It is so named because of its connection with the problem of assigning n objects randomly to N cells. Let $\eta_1, \dots, \eta_N$ be random variables that are, for example, the sizes of components of a graph. If there are independent random variables $\xi_1, \dots, \xi_N$ so that the joint distribution of $\eta_1, \dots, \eta_N$ for any integers $k_1, \dots, k_N$ can be written as

$$\mathbf{P}\{\eta_1 = k_1, \dots, \eta_N = k_N\} = \mathbf{P}\{\xi_1 = k_1, \dots, \xi_N = k_N \mid \xi_1 + \cdots + \xi_N = n\},$$

where n is a positive integer, then we say that $\eta_1, \dots, \eta_N$ satisfy the generalized scheme of allocation with parameters n and N and independent random variables $\xi_1, \dots, \xi_N$.

Graph evolution is the random process of sequentially adding new edges to a graph. For many classes of random graphs with n labeled vertices and T edges, the parameter $\theta = 2T/n$ plays a role of time in the process; various graph properties often change abruptly at the critical point $\theta = 1$. Graph evolution is the most fascinating object in the theory of random graphs, and it appears that it is well suited to the generalized scheme. We will show that applying generalized schemes makes it possible to analyze random graphs at different stages of their evolution and to obtain limit distributions in those cases in which only properties similar to the law of large numbers have been proved.

The theory of random equations in finite fields is shared by probability, combinatorics, and algebra. In this book, we will consider systems of linear equations in GF(2) with random coefficients. The matrix of such a system corresponds to a random graph or hypergraph; therefore, results on random graphs help to study these systems. We are sure that this application alone justifies developing the theory of random graphs.

The theory of random permutations is a well-developed branch of probabilistic combinatorics. Although Goncharov has investigated the cycle structure of a random permutation in great detail, there is still great interest in this area. We will fully describe the asymptotic behavior of $\mathbf{P}\{\nu_n = k\}$ for the total number ν_n of cycles in a random permutation for all possible behaviors of the parameters n and $k = k(n)$ as $n \to \infty$. We will also give some of the asymptotic results for the number of solutions of the equation $X^d = e$, where an unknown $X \in S_n$, d is a fixed positive integer, and e is the identity of the group S_n.

Although the generalized scheme of allocation cannot be applied to nonequiprobable graphs, we present some results in this situation by using the method of moments. The statistical applications of nonequiprobable graphs call for the development of regular methods of analyzing these structures.

The book consists of five chapters. Chapter 1 describes the generalized scheme of allocation and its applications to a random forest of nonrooted trees, a random graph consisting of unicyclic components, and a random graph with a mixture of trees and unicyclic components. In Chapter 2, these results are applied to the study of the evolution of random graphs. Chapter 3 is devoted to systems of random linear equations in GF(2). Much of this branch of probabilistic combinatorics is the work of Russian mathematicians; this is the first English-language presentation of many of the results. Random permutations are considered in Chapter 4, and Chapter 5 contains some results on permutation equations of the form $X^d = e$.

Most results presented in this book derive from work done over the past fifteen years; notes and references can be found in the last section of each chapter. (It is, of course, impossible to give a complete list in each particular area.) In addition to articles used in the text, the summary sections of all chapters include references to papers on related topics, especially those in which the same results were obtained by other methods.

We assume that the reader is familiar with basic combinatorics. This book should be accessible to those who have completed standard courses of mathematical analysis and probability theory. Section 1.1 includes a list of pertinent results from probability.

This book continues in the tradition of *Random Mappings* [78] and differs from other treatments of random graphs in the systematic use of the generalized scheme of allocation. We hope that the chapter on systems of random linear equations in GF(2) will be of interest to a broad audience. I wish to express my sincere appreciation to G.-C. Rota, who encouraged me to write this book for the *Encyclopedia of Mathematics* series, even though there are already several excellent books on random graphs.

My greatest concern is writing the book in English. I am indebted to the editors who have brought the text to an acceptable form. It is apparent that no amount of editing can erase the heavy Russian accent of my written English, so my special thanks go to those readers who will not be deterred by the language of the book.

I greatly appreciate the support I received from my colleagues at the Steklov Mathematical Institute while I wrote this book.