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**Duality and perturbation
methods in critical point
theory**



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CONTENTS

<i>Preface</i>	xi
<i>Introduction</i>	xiii
Chapter 1. Lipschitz and Smooth Perturbed Minimization Principles	1
1.1 Ekeland's variational principle	1
1.2 Constrained minimization and global critical points	5
1.3 A non-homogeneous elliptic equation involving the critical Sobolev exponent	8
1.4 Weak sub-solutions and super-solutions	12
1.5 A minimization principle with quadratic perturbations	14
1.6 The Hartree-Fock equations for Coulomb systems	17
1.7 A general perturbed minimization principle	21
1.8 Perturbed minimization and differentiability	26
1.9 Infinite dimensional Hamilton-Jacobi equations	29
Notes and Comments	33
Chapter 2. Linear and Plurisubharmonic Perturbed Minimization Principles	34
2.1 A minimization principle with linear perturbations	34
2.2 Weak*- H_δ subsets of dual Banach spaces	35
2.3 Generic results for non-homogeneous elliptic equations	39
2.4 Analytic martingales and optimization in L^p ($0 < p \leq 1$)	41
2.5 A variational characterization of the plurisubharmonic envelope of a function	43

2.6	A plurisubharmonically perturbed minimization principle	45
	Notes and Comments	50
Chapter 3. The Classical Min-Max Theorem		52
3.1	Homotopy-stable families	52
3.2	The basic deformation lemma for C^1 -manifolds	55
3.3	The standard (PS) condition and a subcritical elliptic boundary value problem	58
3.4	The (PS) condition along a min-maxing sequence of sets and a resonant boundary value problem	61
	Notes and Comments	66
Chapter 4. A Strong Form of the Min-Max Principle		67
4.1	Duality and the location of critical points	67
4.2	The (PS) condition around a dual set and a relaxed boundary condition	72
4.3	The forced pendulum	74
4.4	The structure of the critical set and the existence of good paths	76
4.5	Multiplicity results of Ljusternik-Schnirelmann type	77
4.6	Functionals with a common critical level and multiplicity of non-radial solutions on the annulus	80
	Notes and Comments	89
Chapter 5. Relaxed Boundary Conditions in the Presence of a Dual Set		90
5.1	Another relaxation of the boundary condition	90
5.2	The <i>relaxed</i> mountain pass and saddle point theorems	95
5.3	The forced double pendulum	97
5.4	A <i>relaxed</i> boundary condition in the presence of linking	104
5.5	Second order systems	106
5.6	Critical points in the presence of splitting	108
5.7	Multiple solutions for a boundary value problem	110
5.8	Multiple solutions for a nonlinear eigenvalue problem	111
	Notes and Comments	113
Chapter 6. The Critical Set in the Mountain Pass Theorem		114
6.1	Saddle points of mountain pass type	115
6.2	A theorem of G. Fang	120

6.3	A bifurcation problem	126
	Notes and Comments	127
Chapter 7. Group Actions and Multiplicity of Critical Points		128
7.1	The equivariant min-max principles	128
7.2	Indices associated to a group and multiplicity results	131
7.3	The \mathbf{Z}_2 -index of Krasnoselski	140
7.4	A <i>relaxed</i> \mathbf{Z}_2 -symmetric mountain pass theorem	141
7.5	Multiple solutions in the presence of symmetry	143
7.6	The mod-2 genus	145
	Notes and Comments	146
Chapter 8. The Palais-Smale Condition Around a Dual Set – Examples		147
8.1	A positive solution for an elliptic boundary value problem involving the critical Sobolev exponent	147
8.2	A sign-changing solution for an elliptic boundary value problem involving the critical Sobolev exponent	151
8.3	A second solution for a nonhomogeneous elliptic equation involving the critical Sobolev exponent	158
	Notes and Comments	165
Chapter 9. Morse Indices of Min-Max Critical Points – The Non Degenerate Case		166
9.1	Homotopic, cohomotopic and homological families	166
9.2	The Morse Lemma	169
9.3	The transformation of Lazer-Solimini	170
9.4	Morse indices in the saddle point theorem	176
9.5	An asymptotically linear non-resonant boundary value problem	176
9.6	An asymptotically linear resonant problem of Landesman-Lazer type	178
9.7	An asymptotically linear strongly resonant boundary value problem	181
	Notes and Comments	182
Chapter 10. Morse Indices of Min-Max Critical Points – The Degenerate Case		183
10.1	Multiplicity results for critical points with a given Morse index	183

10.2	Restrictions of a homotopy-stable class to a neighborhood	187
10.3	The Marino–Prodi perturbation method	189
10.4	The reduction to the non-degenerate case	193
10.5	Application to some classical variational settings	199
	Notes and Comments	203
Chapter 11. Morse-type Information on Palais-Smale Sequences		204
11.1	Palais-Smale sequences with second order conditions– the homotopic case	205
11.2	Multiple solutions for the Hartree-Fock equations	215
11.3	Palais-Smale sequences with second order conditions– the cohomotopic case	219
	Notes and Comments	223
<i>Appendices by David Robinson</i>		
A	Relevant function spaces and inequalities	224
B	Variational formulations of some boundary value problems	233
C	The blowing-up of singularities	236
D	Elements of degree theory	238
E	Basic properties of martingales	243
	<i>References</i>	246
	<i>Index</i>	254

INTRODUCTION

Eigenvectors, geodesics, minimal surfaces, harmonic maps, conformal metrics with prescribed curvature, subharmonics of Hamiltonian systems, solutions of semilinear elliptic partial differential equations and Yang-Mills fields are all critical points of some functional on an appropriate manifold. This is not surprising since many of the laws of mathematics and physics can be formulated in terms of *extremum principles*.

Finding such points by minimization is as ancient as the least action principle of Fermat and Maupertuis, and the calculus of variations has been an active field of mathematics for almost three centuries. For more general, unstable extrema, the methods have a more recent history. Two, not unrelated, theories are available for dealing with the existence of such points: *Morse theory* and *the min-max* methods (or the calculus of variations in the large) introduced by G. Birkhoff and later developed by Ljusternik and Schnirelmann in the first half of this century. Currently, both theories are being actively refined and extended in order to overcome the limitations to their applicability in the theory of partial differential equations: limitations induced by the infinite dimensional nature of the problems and by the prohibitive regularity and non-degeneracy conditions that are not satisfied by present-day variational problems.

In this monograph, we will be concerned with the old problems of existence, location, multiplicity and structure of critical points via the methods of the calculus of variations in the large, but we shall present a new point of view that might help in dealing with some of the difficulties mentioned above. Actually, we shall build upon the well known min-max methods that are presently used in non-linear differential equations and

which are well documented in the lecture notes of Rabinowitz [R 1] and the recent books of Ekeland [Ek 2], Mawhin-Willem [M-W] and Struwe [St]. However, unlike the above mentioned monographs, our emphasis will be on trying to develop, in a systematic way, a “general theory” for variationally generated critical points, that can be applied in a variety of situations, under a minimal set of hypotheses. But our real goal is to advertise a whole array of newly discovered *duality and perturbation methods* that are instrumental in carrying out the refinements of that theory.

One of the central questions we address is the following: what can one say about the *critical structure* of a functional defined on an infinite dimensional domain without imposing the usual compactness conditions *à la Palais-Smale*, or the non-degeneracy conditions *à la Fredholm*? We shall present various new variational principles which, besides ensuring the existence of critical points under minimal hypotheses, give valuable information about their location, their multiplicity and their structure. These principles will yield most of the classical results. However we shall emphasize those applications where the standard methods do not apply but the ones offered in this monograph do.

Here is an overview of our approach. Consider a C^1 -functional φ on a smooth infinite dimensional manifold X . If φ is bounded below on X , then one can try to find a critical point at the level $c = \inf_X \varphi$ by checking whether the infimum is attained. If φ is indefinite or if one is looking for *unstable* critical points, one then considers a family \mathcal{F} of compact subsets of X that is stable under a certain class of homotopies and then shows that φ has a critical point at the level

$$c = c(\varphi, \mathcal{F}) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x).$$

In both cases, one can easily find an *almost critical sequence* at the level c ; that is a sequence $(x_n)_n$ in X satisfying:

$$\lim_n \varphi(x_n) = c \quad \text{and} \quad \lim_n \|d\varphi(x_n)\| = 0. \quad (1)$$

The main problem of existence reduces to proving that such a sequence is convergent. This is usually where the hard analysis is needed. Any function that possesses such a property is said to satisfy the *Palais-Smale condition at the level c* (in short, $(PS)_c$). However, various interesting functionals originating in partial differential equations and in differential geometry do not satisfy such a property, or they may only satisfy it for certain levels c . These problems usually occur in situations involving the critical exponent in the Sobolev embedding theorems, or in the cases where *scale or gauge invariance* requirements give rise to non-compact group actions.

Our main purpose in this direction is to try to find almost critical sequences for φ that possess some extra properties which might help to ensure their convergence. Here is the first idea that comes to mind: one can find an almost critical sequence that is arbitrarily close to any *min-maxing sequence* $(A_n)_n$ in \mathcal{F} . That is, if $(A_n)_n$ in \mathcal{F} is such that $\lim_n \max_{A_n} \varphi = c$, then one can find a sequence $(x_n)_n$ that satisfies (1) as well as

$$\lim_n \text{dist}(x_n, A_n) = 0. \quad (2)$$

What is needed then, is a *Palais-Smale condition along one min-maxing sequence* $(A_n)_n$ in \mathcal{F} . As we shall see in Chapter 3, this weakening of the Palais-Smale condition turns out to be relevant for the solution of a *resonance problem*.

Another point of view consists of considering a family \mathcal{F}^* of closed sets that is *dual to* \mathcal{F} (i.e., satisfying $F \cap A \neq \emptyset$ for all $F \in \mathcal{F}^*$ and $A \in \mathcal{F}$) and such that

$$\sup_{F \in \mathcal{F}^*} \inf_{x \in F} \varphi(x) = \inf_{A \in \mathcal{F}} \max_{x \in A} \varphi(x) = c.$$

One of the main results of this monograph (Theorem 4.5) shows that one can then construct an almost critical sequence that is also arbitrarily close to any *max-mining sequence* $(F_n)_n$ in \mathcal{F}^* . In other words, if $(F_n)_n$ in \mathcal{F}^* is such that $\lim_n \inf_{F_n} \varphi = c$, then one can find a sequence $(x_n)_n$ that satisfies (1), (2) and

$$\lim_n \text{dist}(x_n, F_n) = 0. \quad (3)$$

There are three important features to this new min-max principle. First, we could expect that the topology, which provides the dual families, can sometimes help the analysis involved in proving the convergence of such pseudo-critical sequences. We shall see in Chapter 8, that in various examples, one can indeed push back the threshold of non-compactness by ensuring that an almost critical sequence is arbitrarily close to a certain dual set.

Another aspect of that refinement is that it helps relax the boundary conditions: Indeed, the homotopies that preserve the elements of the family \mathcal{F} are usually required to leave a certain boundary B invariant. In that case, the standard existence results require that $\sup \varphi(B) < c = c(\varphi, \mathcal{F})$. In Chapters 4 and 5, we exhibit situations where the boundary condition can be relaxed to $\sup \varphi(B) = c$, especially in the presence of dual sets.

The third and most important feature of this principle is that it locates *critical points* whereas the classical min-max principle only identifies

possible *critical levels*. Indeed, as mentioned in (3) above, we could find, under adequate compactness conditions, a critical point on any dual set F . The largest dual set is clearly $\{\varphi \geq c\}$. The trick is to find proper subsets F of $\{\varphi \geq c\}$ that still intersect all the members in \mathcal{F} , since the smaller F is, the more information we have about the critical points it contains. By appropriate choices of dual sets, we manage to prove various old and new results concerning the multiplicity and the Morse indices of critical points generated by this procedure. In Chapter 6, we use this method to classify the critical points generated by the mountain pass theorem in a setting that is not covered by Morse theory. In Chapter 7, we show how this same information about the location of critical points on various dual sets gives easy proofs of the standard multiplicity results, while it leads naturally to new and unexpected ones. In Chapter 9, it helps us relate the Morse index of a critical point to the *topological dimension* (homotopic, cohomotopic or homological) of the family \mathcal{F} used to obtain it in the case of non-degenerate C^2 -functionals on Riemannian manifolds. In Chapter 10, we study the degenerate case and find that the same principle gives new types of multiplicity results: these concern the size of sets of critical points having a common estimate on their Morse indices.

Let us return to the possibility of “improving” the Palais-Smale sequences in order to ensure their convergence. Suppose now that φ is a C^2 -functional. One can then try to get an almost critical sequence $(x_n)_n$ with some information about the second derivatives $d^2\varphi(x_n)$. For example, this can be done in minimization problems: one can then construct an almost critical sequence $(x_n)_n$ that is minimizing and that satisfies for each $n \in \mathbb{N}$:

$$\langle d^2\varphi(x_n)v, v \rangle \geq -\|v\|^2/n \quad \text{for any } v \in X. \quad (4)$$

One way to obtain such sequences consists of establishing “perturbed variational principles”. (A typical example would be Ekeland’s Theorem). If, due to the lack of compactness, we cannot find critical points for a given function, can one then perturb it so that the new functional has critical points of the kind that is expected for the original one. This type of result is much stronger than finding almost critical points for the original function. Indeed, if the perturbation is C^2 -small, then knowing that the almost critical point is a true critical point for the new functional allows us to use Morse theory and therefore to get some information about the Hessian of the original function at that point.

For minimization problems, several results of this kind have recently been established with various degrees of generality. In Chapter 1, we

include a general variational principle where the perturbations can be taken to be as smooth as the norm on the domain of the functional under study. As an illustration, we give a result of P. L. Lions about the solutions of the *Hartree-Fock equations* as well as an application to the existence of *viscosity solutions* for first-order *Hamilton-Jacobi equations*. We also give, in Chapter 2, two recent results of a similar nature: In the first, which can be applied in reflexive Banach spaces, the perturbations can be taken to be linear while the second covers spaces as “topologically and geometrically bad” as L^1 , provided one settles for plurisubharmonic perturbations. This will be used in Chapter 2 to get generic minimization results in the case of critical exponents and non-zero data.

Perturbed variational principles for problems not involving minimization are more involved and one should expect to need *hyperbolic perturbations*. Results of this type are very recent and are not yet in their final form. However, we shall present in Chapter 11, a new method devised by Fang and Ghoussoub, for constructing directly –without establishing the perturbation result– an almost critical sequence with the appropriate second order information, provided one has an additional assumption of Hölder continuity on the first and second derivatives. For example, in the context of the *mountain pass theorem*, we obtain an almost critical sequence $(x_n)_n$ and a sequence of subspaces $(E_n)_n$ of codimension one such that the sequence $(x_n)_n$ satisfies, in addition to (1), (2) and (3) above, the following condition:

$$\langle d^2\varphi(x_n)v, v \rangle \geq -\|v\|^2/n \quad \text{for any } v \in E_n. \quad (5)$$

In other words, for each $n \in \mathbf{N}$, $d^2\varphi(x_n)$ has *at most one* eigenvalue below $-1/n$, which clearly implies that any potential cluster point for $(x_n)_n$ will be a critical point of Morse index at most one. More importantly, and as already noted by P. L. Lions in his study of the Hartree-Fock equations, this additional information is sometimes crucial for proving the convergence of such sequences, especially when the standard Palais-Smale condition is not satisfied. We shall include this example as an application of our results in the homotopic case.

Let us mention that another weakening of the Palais-Smale condition also turned out to be relevant in some geometric problems. It is the *(PS)* condition along a fixed orbit of the *negative gradient flow* of the functional: that is when compactness holds for those pseudo-critical sequences of the form

$$x_n = \sigma(t_n, x) \quad (6)$$

where σ is the solution of the Cauchy problem

$$\dot{\sigma} = -\nabla\varphi(\sigma), \quad \sigma(0, x) = x.$$

We shall not discuss this phenomenon here and we refer the reader to the book of Bahri [Ba 2] for a detailed analysis of several important examples.

Throughout this monograph, we have also emphasized the role of various *local perturbation methods* that are now available and which are interesting in their own right. These techniques consist of changing a problem appropriately to a neighboring one that can be more manageable. Here is a sample: In Theorem 4.5, we perturb the function and the variational setting to reduce the new refined min-max principle to the old one. In Theorem 5.2, we use yet another perturbation to reduce the new relaxed boundary condition to the classical one. In chapter 10, we use the Marino-Prodi perturbation method to change a degenerate case to a non-degenerate one. A method for restricting a homotopy-stable class of sets to submanifolds is implemented, while another kind of perturbation is needed to isolate certain subsets of critical points.

We have tried to make this monograph as self-contained as possible, especially for the part that deals with “the general theory”. It was hard to do the same with the examples and therefore they may seem sketchy at times. However, the necessary results and concepts are included – some without proofs – in an appendix prepared skillfully by David Robinson. This is definitely not a comprehensive study of critical point theory, nor of any subset of the theory of partial differential equations. It is merely a collection of some recently formulated variational principles, followed by some examples justifying their relevance. Our main goal is to make these new methods accessible to non-linear analysts, hoping that more interesting applications will follow. To illustrate these principles, we chose the simplest examples known to us, and often did not include the strongest known versions, in order to avoid the complications that were irrelevant to our discussion.

Before closing this introduction, we should mention that at present, there are at least two other approaches that have proved very successful in dealing with the difficulties of infinite dimensional critical point theory and which are unfortunately missing from this monograph. We have in mind Conley’s and Floer’s approaches to Morse theory ([Co], [Fl 1,2]) and the machinery involving *critical points at infinity* that was mainly developed by A. Bahri [Ba 2]. The reason for these omissions is simply that both of these theories are substantially more involved than the elementary approach adopted in this monograph since they require a much heavier background from algebraic topology and geometry.