1
INTRODUCTION

1.1 (i) Let $s_0 = 1/2$, $s_n = 1/2 + \sum_{j-1}^{n-1} \cos jx$ for $n \geqslant 1$. By writing $s_n = (\sum_{j=-n}^{n} e^{ijx})/2$ and summing geometric series show that $(n+1)^{-1}\sum_{j=0}^{n} s_j \to 0$ as $n \to \infty$ for all $x \not\equiv 0 \bmod 2\pi$, and so

$$0 = 1/2 + \sum_{j=1}^{\infty} \cos jx \text{ in the Cesàro sense.}$$

(ii) Show similarly that, if $x \not\equiv 0 \bmod 2\pi$, then

$$\cot(x/2) = 2\sum_{j=1}^{\infty} \sin jx \text{ in the Cesàro sense.}$$

1.2$^{(-)}$ (i) Suppose $s_r = (-1)^r(2r+1)$ for $r = 0, 1, 2, \ldots$. Show that $t_n = (n+1)^{-1}\sum_{j=0}^{n} s_j$ does not tend to a limit but that $(n+1)^{-1}\sum_{j=0}^{n} t_j$ does. In other words, applying the Cesàro procedure once does not produce a limit, but applying it twice does.

(ii) Give an example of a sequence where applying the Cesàro procedure twice does not produce a limit, but applying the Cesàro procedure three times does.

1.3 Can we improve on Cesàro? In particular can we find $a_{nj} \in \mathbb{C}$ such that

(A) if $s_n \to s$ then $\sum_{j=0}^{n} a_{nj}s_j \to s$,

(B) $\sum_{j=0}^{n} a_{nj}s_j$ converges for every sequence (s_n)?

A little thought suggests that this is too much to hope, but may leave open the question if we replace condition (B) by

(B)′ $\sum_{j=0}^{n} a_{nj}s_j$ converges for every bounded sequence (s_n).

To tackle this question, we first try to find necessary and sufficient conditions on the a_{nj} to make (A) hold. Two conditions are obviously necessary.

(i) By taking $s_m = 1$ and $s_n = 0$ for $n \neq m$, show that if (A) holds then $a_{nm} \to 0$ as $n \to \infty$ for each fixed m.

(ii) By taking $s_n = 1$ for all n show that, if (A) holds, then $\sum_{j=0}^{n} a_{nj} \to 1$ as $n \to \infty$.

1.4 Suppose now that condition (A) of Question 1.3 holds.

(i) Using part (i) of Question 1.3 show that, given any $M \geqslant 0$ and any $\varepsilon > 0$, we can find an N such that $\sum_{j=0}^{n} |a_{nj}| \leqslant \varepsilon$ for all $n \geqslant N$.

(ii) Hence show, using induction and part (ii) of Question 1.3, that we can find integers $0 = N(0) < N(1) < N(2) < \ldots$ such that

$$\sum_{j=0}^{N(r-1)} |a_{N(r)j}| \leqslant 2^{-r},$$

$$\left| \sum_{j=0}^{N(r)} a_{N(r)j} - 1 \right| \leqslant 2^{-r}$$

for all $r \geqslant 1$.

(iii) Now define $s_j = (-1)^r$ for $N(r-1) \leqslant j < N(r)$ $[r \geqslant 1]$ and observe that the s_j form a bounded sequence with $\sum_{j=0}^{N(2r-1)} a_{N(2r-1)j} s_j \to -1$, $\sum_{j=0}^{N(2r)} a_{N(2r)j} s_j \to 1$. Conclude that (A) and (B)′ can not be simultaneously satisfied.

(iv)* Find necessary and sufficient conditions on $a_{nj} \in \mathbb{C}$ for condition (B)′ of Question 1.3 to be satisfied.

1.5 In Question 1.3 we saw that the conditions

(α) $a_{nj} \to 0$ as $n \to \infty$ for each fixed j,

(β) $\sum_{j=0}^{n} a_{nj} \to 1$ as $n \to \infty$,

are necessary for condition (A) to be satisfied. There is a third condition,

(γ) there exists a K with $\sum_{j=0}^{n} |a_{nj}| < K$ for all n,

which is also necessary, as we prove in this question.

(i) Suppose the a_{nj} satisfy condition (A), but the sequence $K(n) = \sum_{j=0}^{n} |a_{nj}|$ is unbounded. By imitating parts (i) and (ii) of Question 1.4, or otherwise, show that we can find integers $0 = N(0) < N(1) < N(2) < \ldots$ such that

$$\sum_{j=0}^{N(r-1)} |a_{N(r)j}| \leqslant 2^{-r},$$

$$\sum_{j=0}^{N(r)} |a_{N(r)j}| \geq 2^{2r}.$$

(ii) Now recall the definition $\operatorname{sgn}\lambda = \lambda^*/|\lambda|$ if $\lambda \neq 0$, $\operatorname{sgn} 0 = 0$ and take $s_j = 2^{-r}\operatorname{sgn}(a_{N(r)j})$ for $N(r-1) \leq j < N(r)$ $[r \geq 1]$. Show that $s_j \to 0$ as $j \to \infty$ but $\sum_{j=0}^{N(r)} a_{N(r)j}s_j \to \infty$ as $r \to \infty$.
Conclude that condition (γ) is necessary for condition (A) to be satisfied.

(iii) By imitating the proof of Lemma 1.4(i), or otherwise, show that the three conditions (α), (β), (γ) are together also sufficient for condition (A) to hold.

(iv) Find necessary and sufficient conditions on $a_{nj} \in \mathbb{C}$ for condition (B) of Question 1.3 to hold.

1.6 (Generalisations)

(i) Suppose $0 = N(0) < N(1) < N(2) < \dots$ and $a_{nj} \in \mathbb{C}$ $[0 \leq j \leq N(r),\ r \geq 1]$. Show that the condition,

(A) if $s_n \to s$ then $\sum_{j=0}^{N(n)} a_{nj}s_j \to s$,

holds if and only if

(α) $a_{nj} \to 0$ as $n \to \infty$ for each fixed j,

(β) $\sum_{j=0}^{N(n)} a_{nj} \to 1$,

(γ) there exists a K such that $\sum_{j=0}^{N(n)} |a_{nj}| < K$.

(We could repeat the earlier proofs with trivial changes. Another method is to define

$$b_{nj} = a_{N(r-1)j} \text{ for } 0 \leq j \leq N(r-1) \leq n < N(r)$$
$$b_{nj} = 0 \text{ for } N(r-1) < j \leq n < N(r)$$

and apply the result of Question 1.5 to the b_{nj}.)

(ii) In fact many classical techniques involve replacing the sum $\sum_{j=0}^{N(n)} a_{nj}s_j$ in (A) by an infinite sum $\sum_{j=0}^{\infty} a_{nj}s_j$. (We shall see an important instance of such a technique in Chapter 27.) Suppose now that $a_{nj} \in \mathbb{C}$ for all $j, n \geq 0$. Show using the ideas of Question 1.5, or otherwise, that, if $\sum_{j=0}^{\infty} a_{nj}s_j$ converges whenever $s_j \to 0$, then $\sum_{j=0}^{\infty} |a_{nj}|$ converges.

(iii) Now show that, if $a_{nj} \in \mathbb{C}$ for all $j, n \geq 0$, then the condition

(A)′ if $s_n \to s$ as $n \to \infty$, then $\sum_{j=0}^{\infty} a_{nj}s_j$ converges for each fixed n and $\sum_{j=0}^{\infty} a_{nj}s_j \to s$ as $n \to \infty$,

holds if and only if

(α)′ $a_{nj} \to 0$ as $n \to \infty$ for each fixed j,

$(\beta)'$ $\sum_{j=0}^{\infty} a_{nj} \to 1$ as $n \to \infty$,

$(\gamma)'$ there exists a K such that $\sum_{j=0}^{\infty} |a_{nj}| \leqslant K$ for each n.

(Again we could repeat earlier proofs with minor changes. Alternatively we could construct $0 = N(0) < N(1) < \ldots$ in such a way that $\sum_{j=N(n)+1}^{\infty} |a_{nj}| \leqslant 2^{-n}$, and apply part (i).)

This result is due to Toeplitz. From now on we shall refer to an array (a_{nj}) as *regular* if it satisfies condition (A)$'$.

(iv) Show that if (a_{nj}) is regular, then there exists a bounded sequence s_j such that $\sum_{j=0}^{\infty} a_{nj}s_j$ does not tend to a limit as $n \to \infty$.

(v)* Find necessary and sufficient conditions on $a_{nj} \in \mathbb{C}$ for the following to hold:

(A)$''$ if $s_n \to s$ as $n \to \infty$, then $\sum_{j=0}^{\infty} a_{nj}s_j$ converges for each fixed n and $\sum_{j=0}^{\infty} a_{nj}s_j$ tends to a limit as $n \to \infty$.

1.7 (i) Suppose s_n is a sequence in $\mathbb{C}$ which fails to converge as $n \to \infty$ and let $\gamma \in \mathbb{C}$. Show that either

(I) We can find $n(1) < n(2) < \ldots$ such that $|s_{n(j)}| \to \infty$, or
(II) we can find $\alpha, \beta \in \mathbb{C}$ and $n(1) < n(2) < \ldots$ such that $\alpha \neq \beta$ and $s_{n(2r+1)} \to \alpha$, $s_{n(2r)} \to \beta$.

(ii) In case (II) show that we can find $\lambda, \mu \in \mathbb{C}$ such that $\lambda + \mu = 1$ and $\lambda\alpha + \mu\beta = \gamma$. By setting $a_{rn(2r)} = \mu$, $a_{rn(2r+1)} = \lambda$, $\alpha_{rj} = 0$ otherwise, show that there is a regular array (a_{rj}) such that $\sum_{j=0}^{\infty} a_{rj}s_j \to \gamma$ as $r \to \infty$.

(iii) Show that the conclusion of (ii) holds in case (I) also. Conclude that, given any non convergent sequence s_n and any $\gamma \in \mathbb{C}$, we can find a regular array (a_{rj}) such that $\sum_{j=0}^{\infty} a_{rj}s_j \to \gamma$ as $r \to \infty$.

(iv) For which sequences s_n can we find a regular array (a_{rj}) such that $|\sum_{j=0}^{\infty} a_{rj}s_j| \to \infty$?

1.8 Among the arrays (a_{nj}) which are regular in the sense of Question 1.6 there are some which are more natural than others.

(i) Show that the following two conditions on a regular array (a_{nj}) are equivalent.

(A) If s_j is a sequence with $M \geqslant s_j \geqslant 0$ for all j and $\lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj}s_j$ exists then $\lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj}s_j \geqslant 0$.

(B) There exists a regular array (b_{nj}) such that $b_{nj} \geqslant 0$ for all $n, j \geqslant 0$ and $\sum_{j=0}^{\infty} |a_{nj} - b_{nj}| \to 0$ as $n \to \infty$.

(ii) Let us call a regular array (a_{nj}) *positive* if $a_{nj} \geqslant 0$ for all $n, j \geqslant 0$. Show that for any positive regular array (a_{nj}) and any real sequence s:

$$\liminf_{j\to\infty} s_j \leqslant \liminf_{n\to\infty} \sum_{j=1}^{\infty} a_{nj}s_j \leqslant \limsup_{n\to\infty} \sum_{j=1}^{\infty} a_{nj}s_j \leqslant \limsup_{j\to\infty} s_j.$$

Show also that for any real sequence s_j and any s with

$$\liminf_{j\to\infty} s_j \leqslant s \leqslant \limsup_{j\to\infty} s_j$$

there exists a positive regular array (a_{nj}) such that $\sum_{j=0}^{\infty} a_{nj}s_j \to s$ as $n \to \infty$.

1.9 There is another natural condition on regular arrays. To introduce it, we have recourse to the ideas of linear algebra. Let us write $\boldsymbol{v}$ for the sequence of complex numbers who jth term is v_j and, if $\boldsymbol{u}$, $\boldsymbol{v}$ are sequences, let us write $\lambda\boldsymbol{u} + \mu\boldsymbol{v}$ for the sequence whose jth term is $\lambda u_j + \mu v_j$. Let V be the collection of sequences such that $\lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj}v_j$ exists. If $\boldsymbol{v} \in V$, we write $T\boldsymbol{v} = \lim_{n\to\infty} \sum_{j=0}^{\infty} a_{nj}v_j$.

(i) Show that, if $\boldsymbol{u}, \boldsymbol{v} \in V$ and $\lambda, \mu \in \mathbb{C}$, then $\lambda\boldsymbol{u} + \mu\boldsymbol{v} \in V$ and $T(\lambda\boldsymbol{u} + \mu\boldsymbol{v}) = \lambda T\boldsymbol{u} + \mu T\boldsymbol{v}$.

(ii) If $\boldsymbol{v}$ is a sequence, let us write $S\boldsymbol{v}$ for the sequence whose jth term is v_{j+1}. Show that $S(\lambda\boldsymbol{u} + \mu\boldsymbol{v}) = \lambda S\boldsymbol{u} + \mu S\boldsymbol{v}$. Show that, if $u_j \to u$, then $u_{j+1} \to u$ as $j \to \infty$.

(iii) In view of (ii), it is natural to demand that the regular array (a_{nj}) should be *translation consistent* in the sense that, if $\boldsymbol{v} \in V$, then $S\boldsymbol{v} \in V$ and $TS\boldsymbol{v} = T\boldsymbol{v}$.

Now suppose that (a_{nj}) is indeed regular and translation consistent and that $v_j = z^j$ for some $z \in \mathbb{C}$. Show that, if $\boldsymbol{v} \in V$, then $T\boldsymbol{v} = TS\boldsymbol{v} = zT\boldsymbol{v}$, and deduce that, if $z \neq 1$, then $T\boldsymbol{v} = 0$. What happens if $z = 1$?

(iv) With the same hypotheses on (a_{nj}) and with the same definition of V show that, if $u_j = \sum_{r=0}^{j} v_r$, then

(a) $\boldsymbol{u} \in V$ if and only if $\boldsymbol{v} \in V$ and $z \neq 1$,
(b) if $\boldsymbol{u} \in V$, then $T\boldsymbol{u} = (1 - z)^{-1}$.

Taking $z = -1$ we obtain a formal version of the argument, which goes back at least as far as Leibnitz, that if the sum S of $1 - 1 + 1 - \ldots$ has any meaning then we must have $S = 1/2$.

(v) Suppose now that (a_{nj}) is also positive. By taking $z = e^{i\theta}$ and taking real and imaginary parts (explain carefully why this is possible), show that if the formula $u_j = \sum_{r=0}^{j} e^{ir\theta}$ defines a sequence $\boldsymbol{u} \in V$ for some $\theta \in \mathbb{R}$, then writing $b_j = 1/2 + \sum_{r=0}^{j} \cos r\theta$, $c_j = \sum_{r=0}^{j} \sin r\theta$ we have $\boldsymbol{b}, \boldsymbol{c} \in V$ and

$$Tb = 0, \quad 2Tc = \cot \theta/2.$$

Comment on the relation of these results to those of Question 1.1.

(vi) Suppose (a_{nj}) is a regular, translation consistent array. Let $u_j = (-1)^j j$. Show that, if $u \in V$, then $Tu = -1/4$. Let $w_j = (-1)^j j^2$. Assuming that $w \in V$, compute Tw.

(vii) Let (a_{nj}) be a regular array. Show that (a_{nj}) is translation consistent if and only if $\sum_{j=0}^{\infty} a_{njk+l} \to k^{-1}$ for each integer $k \geq 1$ and each integer $l \geq 0$. (You may wish to consider the sequence s_r given be $s_{jk} = 1$ for $j \geq 0$, $s_r = 0$ otherwise.)

(viii) Show that the Cesàro array given by $a_{nj} = (n+1)^{-1}$ for $0 \leq j \leq n$, $a_{nj} = 0$ otherwise, is translation consistent.

(ix) Let $s_j = (-1)^r$ for $2^r - 1 \leq j < 2^{r+1} - 1$ $[r \geq 0]$. Show that given any λ with $-1 \leq \lambda \leq 1$ we can find a regular, positive, translation consistent array (a_{nj}) such that $\sum_{j=0}^{\infty} a_{nj} s_j \to \lambda$ as $n \to \infty$.

1.10 (i) We start with a preliminary calculation. Observe that, if $f(n) \leq f(n+1) \leq \ldots \leq f(n+m)$, $0 \leq l < k$ and $l + rk \leq m$, then

$$\sum_{j=0}^{r-1} f(n+jk) \leq \sum_{j=0}^{r-1} f(n+jk+l) \leq \sum_{j=1}^{r} f(n+jk).$$

Now suppose $g: \mathbb{N} \to \mathbb{R}$ satisfies $g(r) \geq 0$ for all r,

$$g(0) \leq g(1) \leq \ldots \leq g(M), \quad g(M) \geq g(M+1) \geq \ldots,$$

and that $\sum_{r=0}^{\infty} g(r)$ converges. By using the idea of the first sentence show that

$$\left| \sum_{j=0}^{\infty} g(kj) - \sum_{j=0}^{\infty} g(kj+l) \right| \leq 4g(M)$$

for all $0 \leq l < k$.

(ii) Next we have a trivial but, to my mind, interesting observation. Let $X_0, X_1, \ldots$ be random variables taking non negative integer values and such that $P(X_n = r) \to 0$ as $n \to \infty$ for each fixed r. Show that, if $a_{nr} = P(X_n = r)$, then (a_{nr}) is a regular positive array. It is thus possible that interesting random variables such as binomial (coin tossing) and Poisson will give rise to interesting arrays.

(iii) Let $0 < p < 1$ and let

$$a_{nr} = \binom{n}{r} p^r (1-p)^{n-r}$$

for $0 \leq r \leq n$, $a_{nr} = 0$ otherwise. Show that a_{nr} form a regular positive translation consistent array. (You may find Question 1.9 (vii) and part (i) of this question useful.) Show that $\sum_{r=0}^{n} a_{nr} z^r$ tends to a limit as $n \to \infty$ if and only if $|1 + (z-1)p| < 1$ or $z = 1$. Find a value of p such that $\sum_{r=0}^{n} a_{nr}(-2)^r$ converges. Returning to the ideas of Question 1.9(iv), we may say that, if the sum $1-2+4-\dots$ has any meaning, it must be 1/3 and we have found a method for giving meaning to that sum.

(iv) Let $\lambda(n)$ be a sequence of real positive numbers with $\lambda(n)\to\infty$. Let $a_{nr}=\mathrm{e}^{-\lambda(n)}\lambda(n)^r/r!$ for $n, r\geqslant 0$. Show that the a_{nr} form a regular positive translation consistent array. Show that $\sum_{r=0}^{\infty}a_{nr}z^r$ tends to a limit if and only if $\operatorname{Re} z<1$ or $z=1$.

(v) If $p\in\mathbb{C}$ and $p\neq 0, 1$ set

$$a_{nr}=\binom{n}{r}p^r(1-p)^{n-r}$$

for $0\leqslant r\leqslant n$, $a_{nr}=0$ otherwise. For what values of p is the array (a_{nr}) regular?

(vi) Let $\lambda(n)$ be a sequence of real positive numbers with $\lambda(n)\to\infty$. Let $\alpha\in\mathbb{C}$, $\alpha\neq 0$ and set $a_{nr}=\mathrm{e}^{-\alpha\lambda(n)}(\alpha\lambda(n))^r/r!$ for $n, r\geqslant 0$. For what values of α is (a_{nr}) regular? (Part (iii) goes back to Euler, part (iv) to Borel.)

1.11 Question 1.7 to 1.9 and Question 1.10(iv) suggest that our attempt to generalise the notion of a limit via regular arrays is at once too broad and too narrow. The attempt is too broad because many regular arrays do not give 'natural' results, and too narrow because there may be processes which give 'natural' results, but do not depend on arrays. Let us try another approach using the ideas of Question 1.9.

Let U be the space of all sequences of complex numbers and write $\boldsymbol{u}$ for the sequence of complex numbers $u_0, u_1\dots$. Let B be the set of all bounded sequences and C the set of all convergent sequences. Let $S\colon U\to U$ be given by $S\boldsymbol{u}=\boldsymbol{u}$ where $w_i=u_{i+1}$ $[i\geqslant 0]$.

(i) Show that U is a vector space. Show that B is a subspace of U and C subspace of B. Show that $S\colon U\to U$ is linear and that S is surjective. Is S injective? Show that $S(B)=B$, $S(C)=C$.

(ii) Let V be a subspace of U and $L\colon V\to\mathbb{C}$ a linear map. We say that L is *a generalised limit* if

(A) $C\subseteq V$ and $L\boldsymbol{u}=\lim_{n\to\infty}u_n$ for all $\boldsymbol{u}\in C$,
(B) $S(V)=V$ and $LS\boldsymbol{u}=L\boldsymbol{u}$ for all $\boldsymbol{u}\in V$.

If we say that L preserves positivity if, in addition,

(C) if $\boldsymbol{u}\in V$ and u_n is real and non negative for all n, then $L\boldsymbol{u}$ is real and non negative.

Show that, if (a_{nj}) is regular array, and we write

$$V=\left\{\boldsymbol{u}\colon \lim_{n\to\infty}\sum_{j=0}^{\infty}a_{nj}u_j \text{ exists}\right\},\quad L\boldsymbol{u}=\lim_{n\to\infty}\sum_{j=0}^{\infty}a_{nj}u_j,$$

then L is a generalised limit if and only if (a_{nj}) is translation consistent. Show that L is a generalised limit which preserves positivity if and only if there is a positive translation consistent array (b_{nj}) such that $\sum_{j=1}^{\infty}|a_{nj}-b_{nj}|\to 0$.

(iii) Let V be the set of $\boldsymbol{u}$ such that $\sum_{j=0}^{\infty}u_jx^j/j!$ converges for all $x \geqslant 0$ and $e^{-x}\sum_{j=0}^{\infty}u_jx^j/j!$ tends to a limit $L(\boldsymbol{u})$, say, as $x\to\infty$. Using Question 1.10(iv), or otherwise, show that L is a generalised limit which preserves positivity.

(iv) Let $f_j\colon[0,\infty)\to\mathbb{C}$ be a collection of functions such that $\sum_{j=0}^{\infty}|f_j(x)|$ converges for all $x\geqslant 0$. Let V be the set of $\boldsymbol{u}$ such that $\sum_{j=0}^{\infty}u_jf_j(x)$ converges for all $x\geqslant 0$ and $\sum_{j=0}^{\infty}u_jf_j(x)$ tends to a limit $L(\boldsymbol{u})$, say, as $x\to\infty$. Find necessary and sufficient conditions on the f_j for L to be a generalised limit. When does L preserve positivity?

(v) By using the fact that power series can be multiplied term by term within their circles of convergence, show that there exist $b_n\in\mathbb{R}$ such that $\sum_{n=0}^{\infty}b_nx^n=e^x\sin\pi x$ for all $x\in\mathbb{R}$. Let $u_n=n!b_n$. Show that $\boldsymbol{u}\notin V$ where V is the set of part (iii) of this question but that, if, following Question 1.10(iv), we set $a_{nr}=e^{-n}n^r/r!$ then $\sum_{r=0}^{\infty}a_{nr}u_r$ tends to a limit as $n\to\infty$.

(vi) Let L be a generalised limit and V its associated vector space. Let $v_j=z^j$ and $w_j=j$. By recalling the proof in Question 1.9(iii), or otherwise, show that, if $\boldsymbol{v}\in V$, then $L\boldsymbol{v}=0$ or $z=1$. Show also that $\boldsymbol{w}\notin V$. Show that, if L preserves positivity and u_j is a sequence of real numbers with $u_j\to\infty$, then $\boldsymbol{u}\notin V$.

(Thus $V\neq U$. This does not exclude the possibility that we can find a generalised positivity preserving limit L with $V\supseteq B$. The reader is invited to try to decide whether such an L and V can exist, but warned that no answer is possible within the mind set of this book. What we have here is a seventeenth-century problem whose resolution depends on twentieth century ideas from the foundations of mathematics.)

Cambridge University Press
0521438497 - Exercises for Fourier Analysis
T. W. Korner
Excerpt
More information

2

PROOF OF FEJÉR'S THEOREM

2.1$^{(-)}$ By considering the function $f(t) = 0$ for $t \neq 0$, $f(0) = 1$ show that the condition f continuous cannot be dropped in Theorems 2.3 and 2.4.

2.2 If $f: \mathbb{T} \to \mathbb{C}$ is continuous and $g = \operatorname{Re} f$ show that $\hat{g}(r) = (\hat{f}(r) + \hat{f}(-r)^*)/2$. Find the Fourier coefficients of $\operatorname{Im} f$.

2.3 (An alternative proof of Theorem 2.4)

(i) Suppose $f: \mathbb{T} \to \mathbb{R}$ is continuous but $f \neq 0$ and so there exists an $c \in \mathbb{T}$ with $f(c) \neq 0$. By considering $-f$ if necessary, we may suppose $f(c) > 0$. Explain why we can find a $\pi/2 > \delta > 0$ such that $f(t) > f(c)/2$ for $|t - c| < \delta$ and a $K > 0$ such that $|f(t)| \leqslant K$ for all $t \in \mathbb{T}$.

(ii) Show that we can find an $\varepsilon > 0$ so small that $h(t) = \varepsilon + \cos(t - c)$ satisfies $|h(t)| \leqslant 1 - \varepsilon/2$ for $|t - c| > \delta$. Show that there exists an η with $\delta > \eta > 0$ such that $h(t) \geqslant 1 + \varepsilon/2$ for $|t - c| < \eta$. Show that

$$\left|\frac{1}{2\pi}\int_{|t-c|\geqslant\delta} h(t)^n f(t)\,dt\right| \leqslant K(1 - \varepsilon/2)^n \to 0 \quad \text{as } n \to \infty,$$

$$\frac{1}{2\pi}\int_{\delta>|t-c|\geqslant\eta} h(t)^n f(t)\,dt \geqslant 0 \quad \text{for all } n \geqslant 1,$$

and

$$\frac{1}{2\pi}\int_{\eta>|t-c|} h(t)^n f(t)\,dt \geqslant \frac{\eta f(c)}{4\pi}(1 + \varepsilon/2)^n \to \infty \quad \text{as } n \to \infty.$$

Deduce that

$$\frac{1}{2\pi}\int_{\mathbb{T}} h(t)^n f(t)\,dt \to \infty \quad \text{as } n \to \infty.$$

(iii) By first considering the case $n = 1$, show that we can write $h(t)^n = \sum_{j=-n}^{n} a_{nj} e^{ijt}$ for some $a_{nj} \in \mathbb{C}$. Show that

$$\frac{1}{2\pi}\int_{\mathbb{T}} h(t)^n f(t)\,dt = \sum_{j=-n}^{n} a_{nj}\hat{f}(-j)$$

and deduce that, if $\hat{f}(j) = 0$ for all j, then

$$\frac{1}{2\pi}\int_{\mathbb{T}} h(t)^n f(t)\,dt = 0 \quad \text{for all } n.$$

Conclude that, if $f\colon \mathbb{T} \to \mathbb{R}$ is continuous and $\hat{f}(r) = 0$ for all r, then $f = 0$.

(iv) By considering the real and imaginary parts of f, show that, if $f\colon \mathbb{T} \to \mathbb{C}$ is continuous and $\hat{f}(r) = 0$ for all r, then $f = 0$.

2.4 (i) Show that, if P and Q are trigonometric polynomials, then

$$\frac{1}{2\pi}\int_{\mathbb{T}} P(t)Q(mt)\,dt = \frac{1}{2\pi}\int_{\mathbb{T}} P(t)\,dt\,\frac{1}{2\pi}\int_{\mathbb{T}} Q(t)\,dt$$

whenever m is a sufficiently large integer.

(ii) By using Theroem 2.5 to produce a sequence Q_n of trigonometric polynomials with $Q_n \to g$ uniformly, show that, if P is a trigonometric polynomial and g a continuous function, then

$$\frac{1}{2\pi}\int_{\mathbb{T}} P(t)g(mt)\,dt \to \frac{1}{2\pi}\int_{\mathbb{T}} P(t)\,dt\,\frac{1}{2\pi}\int_{\mathbb{T}} g(t)\,dt \quad \text{as } m \to \infty.$$

(iii) Deduce that, if f and g are continuous,

$$\frac{1}{2\pi}\int_{\mathbb{T}} f(t)g(mt)\,dt \to \frac{1}{2\pi}\int_{\mathbb{T}} f(t)\,dt\,\frac{1}{2\pi}\int_{\mathbb{T}} g(t)\,dt \quad \text{as } m \to \infty.$$

(iv) Hence show that, if $f\colon\mathbb{T} \to \mathbb{C}$ is continuous, $\hat{f}(m) \to 0$ and $\hat{f}(-m) \to 0$ as $m \to \infty$. (This is the Riemann Lebesgue lemma proved later as Theorem 52.4.)

2.5 (A Bernstein inequality) If P is a trigonometric polynomial of degree n or less (i.e. $P(t) = \sum_{r=-n}^{n} a_r \exp(irt)$) show that

$$P'(t) = \frac{1}{2\pi}\int_{\mathbb{T}} P(t-y)L_n(y)\,dy$$

where $L_n(y) = -2nK_{n-1}(y)\sin ny$. (Hint: it suffices to verify the simple cases $P(t) = \exp(irt)$ with $-n \leqslant r \leqslant n$.) Hence deduce that

$$|P'(t)| \leqslant 2n \sup_{x\in\mathbb{T}} |P(x)|.$$

(This result can be improved by replacing $2n$ by n. See e.g. Question 43.8.)