Part I · General concepts

1
Hamiltonian dynamics

The equations of motion in classical physics differ considerably depending upon the subject they describe: a particle, an electromagnetic field, or a fluid. However our natural yearning for unification in the description of different phenomena has long since led to the development of universal formalisms. Among these the Lagrangian and Hamiltonian formalisms are the most advanced. This can be explained by the nature of the phenomena discussed. The popularity of each method varied at different stages in the development of physics. Throughout the whole period of advancement of relativistically invariant theories, preference was chiefly given to Lagrangian formalism (this was most conspicuous in field theory and the theory of a continuous medium). To a large extent, it was not before the generalization of the concepts of Hamiltonian formalism and introduction of Poisson's brackets that the Hamiltonian method of analysis was able to compete with the Lagrangian one.

The formation of new ideas and possibilities triggered recently by the discovery of the phenomenon of dynamic stochasticity (or simply, chaos) have brought the methods of Hamiltonian dynamics to the fore. Liouville's theorems on the conservation of phase volume and on the integrability of systems with a complete set of integrals of motion have determined both the formulation of many problems of dynamics and the methods of their study. The Hamiltonian method turned out to be of extreme importance for the theory of stability, which was advanced in this direction by Poincaré. Numerous subsequent studies have shown that Hamiltonian systems (i.e., systems which can be described by Hamiltonian equations of motion) demonstrate fundamental physical differences from other (non-Hamiltonian) systems. This chapter provides the most necessary information on Hamiltonian systems. (Note 1.1)

1.1 Hamiltonian systems

The state of a Hamiltonian system can be described by N generalized momenta $p \equiv (p_1, \ldots, p_N)$ and the same number N generalized coordinates $q \equiv (q_1, \ldots, q_N)$. Here N designates the number of a system's degrees of freedom. The evolution of p and q in time is determined by the equations of motion

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}; \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}; \qquad (i = 1, \ldots, N), \tag{1.1.1}$$

which make sense only together with the Hamiltonian

$$H = H(p, q, t). \tag{1.1.2}$$

Here a dot over a symbol stands for the time t derivatives. The Hamiltonian function (or Hamiltonian) is given in $2N$-dimensional phase space (p, q) and may also be an explicit function of time. Pairs of variables (p_i, q_i) are called canonically conjugate pairs and the equations (1.1.1) are canonical equations.

Time t can also be added to the set of the system's coordinate variables. In order to do this, the phase space of a system should be expanded by way of introduction of one more pair of canonical variables

$$p_0 = -H; \qquad q_0 = t. \tag{1.1.3}$$

Now the Hamiltonian

$$\mathcal{H} = H(p, q, q_0) + p_0 \tag{1.1.4}$$

defines the following equations of motion

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i}; \qquad \dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}; \qquad (i = 0, 1, \ldots, N). \tag{1.1.5}$$

For $i = 1, \ldots, N$ the system (1.1.5) leads to the familiar equations (1.1.1). For $i = 0$ in accordance with (1.1.3) and (1.1.4) we get

$$\begin{aligned} \dot{p}_0 &= -\frac{\partial H(p, q, q_0)}{\partial q_0} = -\frac{\partial H(p, q, t)}{\partial t}; \\ \dot{q}_0 &= \frac{\partial \mathcal{H}}{\partial p_0} = 1. \end{aligned} \tag{1.1.6}$$

The first equation in (1.1.6) signifies the well-known equality

$$\frac{\mathrm{d}H(p, q, t)}{\mathrm{d}t} = \frac{\partial H}{\partial t} \tag{1.1.7}$$

which can be easily verified by means of the equations of motion (1.1.1). The second equation in (1.1.6) reflects the definition of time as one of the coordinates.

Thus instead of an N-dimensional system with the Hamiltonian as a function of time we may consider an $(N+1)$-dimensional system with a time-independent Hamiltonian $\mathscr{H}$. However from the definitions (1.1.3) and (1.1.4) it becomes clear that $\mathscr{H} \equiv 0$. The new momentum $p_0 = -H$ does not bear any additional information. All the properties of a dynamic system can be described in a $(2N+1)$-dimensional phase space $(p_1, \ldots, p_N, q_1, \ldots, q_N, q_0 = t)$. It is therefore sometimes said that a system with the Hamiltonian $H(p, q, t)$ has $N+1/2$ degrees of freedom.

Now let us introduce in a phase space (p, q) an element of phase volume

$$\mathrm{d}\Gamma = \mathrm{d}p_1 \cdots \mathrm{d}p_N \, \mathrm{d}q_1 \cdots \mathrm{d}q_N \equiv \mathrm{d}p \, \mathrm{d}q$$

and a phase volume

$$\Gamma = \int_S \mathrm{d}\Gamma$$

with some hypersurface S as a boundary. Generally speaking, phase volume of a dynamic system is a function of time. However, for Hamiltonian systems phase space is conserved

$$\Gamma_{t_1} = \Gamma_{t_2} \tag{1.1.8}$$

for arbitrary moments of time t_1 and t_2 (Liouville's theorem). This means in particular that phase fluid is incompressible. The property of phase volume conservation has some profound consequences. According to one of them, among all the conceivable trajectories there are none possessing an asymptotically stable equilibrium position (either points or sets of points attracting the trajectory). To put it otherwise, Liouville's theorem rules out the existence of attractors. It is possible to make a similar statement concerning the absence of repellers – the repellent points or sets of points.

The conservation of phase volume holds true not only for Hamiltonian systems. The progress of contemporary nonlinear analysis has led to a generalization of the concept of Hamiltonian systems (see [4]). Let us assume that z_i is a coordinate in $2N$-dimensional phase space, the variables z_i not yet separated into generalized coordinates and generalized momenta. First, the operation of generalized Poisson's brackets should be introduced. For the two arbitrary functions $A(z)$ and $B(z)$, Poisson's brackets are defined by the formula

$$[A, B] = \sum_{i,k=1}^{2N} g_{ik} \frac{\partial A}{\partial z_i} \frac{\partial B}{\partial z_k} \tag{1.1.9}$$

where the tensor $g_{ik} = g_{ik}(z)$ in general depends on the variables. It is also assumed that the following conditions are satisfied:

(a) the bilinearity condition

$$[aA + bB, C] = a[A, C] + b[B, C] \tag{1.1.10}$$

where a and b are constants;

(b) the skew-symmetry condition

$$[A, B] = -[B, A]. \tag{1.1.11}$$

This condition, for example, cannot be satisfied if the phase space dimensionality is uneven;

(c) Leibniz equality

$$[AB, C] = B[A, C] + A[B, C]; \tag{1.1.12}$$

(d) Jacoby's equality

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0. \tag{1.1.13}$$

With the help of Poisson's brackets (1.1.9) we are able to define the tensor g_{ik} as follows

$$g_{ik} = [z_i, z_k], \tag{1.1.14}$$

while the brackets themselves can be expressed in the following way

$$[A, B] = \sum_{i,k=1}^{2N} \frac{\partial A}{\partial z_i} \frac{\partial B}{\partial z_k} [z_i, z_k]. \tag{1.1.15}$$

Now let us consider the arbitrary function $H = H(z)$, which is referred to as the Hamiltonian. A system is called a generalized Hamiltonian system if it can be described by the following equations of motion

$$\dot{z}_i = [z_i, H], \qquad (i = 1, \ldots, 2N). \tag{1.1.16}$$

The variations of any system $A(z)$ with time can be defined with the help of the equations (1.1.14)–(1.1.16):

$$\dot{A} = \sum_{i=1}^{2N} \frac{\partial A}{\partial z_i} \dot{z}_i = \sum_{i=1}^{2N} \frac{\partial A}{\partial z_i} [z_i, H] = [A, H]. \tag{1.1.17}$$

Specifically, if g_{ik} is an identity skew-symmetry matrix,

$$g_{ik} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

equations (1.1.16) are equivalent to equations (1.1.1) for $N = 1$ and $z_1 = q$, $z_2 = p$. For an arbitrary N, the equations (1.1.1) should follow from (1.1.16), if we assume that

$$g_{ik} = \begin{pmatrix} 0 & \hat{1} \\ -\hat{1} & 0 \end{pmatrix}, \tag{1.1.18}$$

where $\hat{1}$ is the identity matrix of the order of N and

$$(z_1, \ldots, z_n) = (q_1, \ldots, q_N);$$

$$(z_{N+1}, \ldots, z_{2N}) = (p_1, \ldots, p_N).$$

The above generalized form of Hamiltonian dynamics will be applied later during an analysis of equations of motion of vector fields.

In the special case of (1.1.18), where the Hamiltonian equations (1.1.1) are true, Poisson's brackets may be presented in the following way

$$[A, B] = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial B}{\partial q_i} \frac{\partial A}{\partial p_i} \right) \tag{1.1.19}$$

and time derivatives (1.1.6) can be presented as

$$\dot{A} = \sum_{i=1}^{N} \left(\frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial H}{\partial q_i} \frac{\partial A}{\partial p_i} \right). \tag{1.1.20}$$

The equations of motion written in the form (1.1.1) or (1.1.16), or equivalent definitions of Poisson's brackets, are fundamental to canonical or Hamiltonian formalism.

1.2 The phase portrait

The system's family of trajectories in phase space comprises its phase portrait. The simplest form of the phase portrait is obtained for $N = 1$. Phase space then is a plane (p, q). One-dimensional motion, for example, may be defined by the Hamiltonian

$$H = \frac{1}{2m} p^2 + V(q), \tag{1.2.1}$$

where $V(q)$ is the potential energy (the potential) of a particle. According to (1.1.7), if the Hamiltonian is not an explicit function of time, then $\dot{H} = 0$ and H is an integral of motion (invariant):

$$H = \text{const} = E, \tag{1.2.2}$$

where E is some fixed value of the invariant. In the case of (1.2.1) the quantity E is the system's total energy. In the example (1.2.1), motion occurs on the surface of constant energy and

$$p = \pm\{2m[E - V(q)]\}^{1/2} = m\dot{q}. \tag{1.2.3}$$

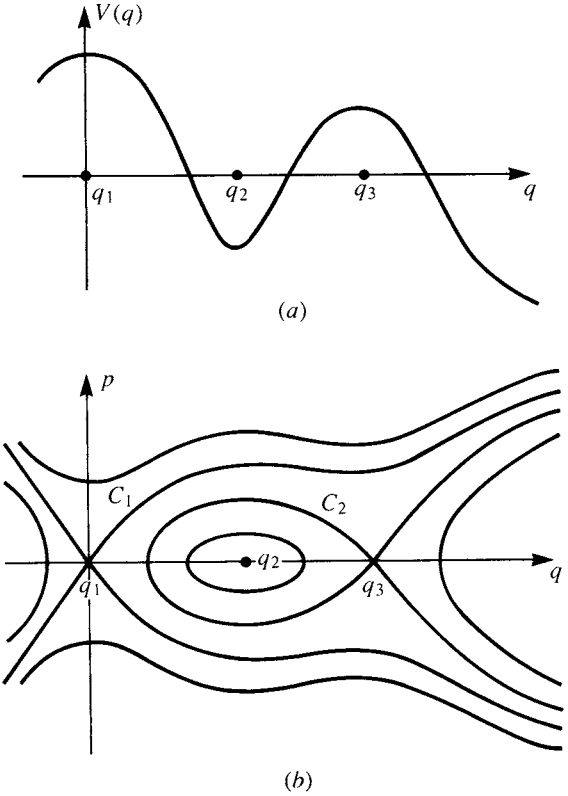
Equation (1.2.3) defines a parametric family of trajectories comprising the phase portrait of the system (Fig. 1.2.1).

If a system's trajectory is localized in a finite region of phase space, the corresponding motion is said to be finite. Otherwise, it is infinite.

Singularities in phase space are defined as the fixed points of the equations of motion, so that they may be derived from the following equations

$$\frac{\partial H}{\partial q_i} = 0; \qquad \frac{\partial H}{\partial p_i} = 0. \tag{1.2.4}$$

Fig. 1.2.1 Potential $V(q)$ (a) and the corresponding phase portrait (b) of one-dimensional motion. The trajectories C_1 and C_2 are separatrices.



In case (1.2.1), these are the points where $p = 0$ and the potential $V(q)$ has an extremum: $V'(q) = 0$.

Figure 1.2.1 shows that these points can be either of the elliptic type (q_2), or of the hyperbolic type (q_1), or saddles. In the neighbourhood of an elliptic point the motion is stable, and the trajectories have the shape of ellipses (Fig. 1.2.2*a*). In the neighbourhood of a saddle the motion is unstable, and the trajectories have the shape of hyperbolas (Fig. 1.2.2*b*). The trajectory passing through a saddle is called a separatrix (trajectories C_1 and C_2 on Fig. 1.2.1). A saddle always has entering and outgoing whiskers of separatrices. (1.2.2*b*).

The absence of limit points and limit quantities such as a limit cycle makes the phase portrait of Hamiltonian systems less diverse. Nevertheless, it does not lessen their complexity. Introduction of dissipative factors and the asymptotically limited trajectories associated with them, makes the system in a way less sensitive to various small perturbations (Note 1.2). This book is almost exclusively dedicated to the properties of the Hamiltonian systems, which are absent in the dissipative case (Note 1.3). Therefore, having made this remark, we do not go into any further details here.

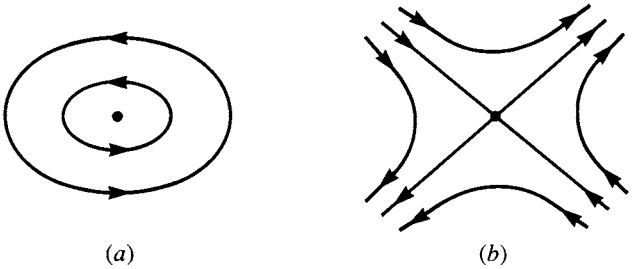
1.3 'Action-angle' variables

For the sake of convenience, we shall first introduce 'action-angle' variables for the case of motion with one degree of freedom (1.2.1). In the case of finite and therefore periodic motion, action I is defined by the expression:

$$I = \frac{1}{2\pi} \oint p \, dq, \tag{1.3.1}$$

where integration is performed over a closed orbit of the system. In the

Fig. 1.2.2 The trajectories in the vicinity of an elliptic (*a*) and a hyperbolic point (*b*).



(*a*) (*b*)

case of infinite but still periodic motion, integration in (1.3.1) should be performed over the whole period. The value

$$S(q, I) = \int^{q} p(q, H)\, \mathrm{d}q \tag{1.3.2}$$

is called truncated action. Both here and in (1.3.1) the expression for $p = p(q, H)$ is given in equation (1.2.3). We should also substitute (1.3.1) for I in (1.3.2) having replaced H by $H(I)$. In this case the value $S(q, I)$ is also a generating function. It enables us to define a new coordinate, the angle:

$$\theta = \frac{\partial S(q, I)}{\partial I}. \tag{1.3.3}$$

The variables (I, θ) make up a canonically conjugate pair, i.e. the Hamiltonian equations of motion will be valid for them:

$$\begin{aligned} \dot{I} &= -\frac{\partial H(I)}{\partial \theta} = 0; \\ \dot{\theta} &= \frac{\partial H(I)}{\partial I} = \frac{\mathrm{d}H(I)}{\mathrm{d}I} \equiv \omega(I). \end{aligned} \tag{1.3.4}$$

Being expressed as a function of the integral of motion H, the action is itself an integral. This is reflected in the first equation in (1.3.4). The second equation defines the frequency of periodic motion $\omega(I)$. In general, it is the function of action and the system's energy $E = H$. The dimensionless parameter

$$\alpha = \frac{I}{\omega}\frac{\mathrm{d}\omega}{\mathrm{d}I} \tag{1.3.5}$$

determines the degree of this dependence. If $\alpha \neq 0$, the oscillations are called nonlinear. Their frequency is a function of energy. Now we can integrate the equations of motion (1.3.4):

$$I = \text{const}; \qquad \theta = \omega(I)t + \theta_0. \tag{1.3.6}$$

That is why the 'action-angle' variables are so useful (although, as we shall see later, this is not their only merit).

For a linear oscillator:

$$H = \tfrac{1}{2}p^2 + \tfrac{1}{2}\omega_0^2 q^2 \tag{1.3.6a}$$

(unit mass is assumed). With the help of definitions (1.3.1)–(1.3.3) we can easily obtain

$$\begin{aligned} q &= (2I/\omega)^{1/2} \cos\theta; \\ p &= (2I\omega)^{1/2} \sin\theta; \\ \theta &= \omega_0 t. \end{aligned} \tag{1.3.7}$$

From (1.3.6a) and (1.3.7) there follows

$$H = \omega_0 I, \tag{1.3.8}$$

i.e., $\omega = \text{const} = \omega_0$ and $\alpha = 0$. This is exactly what determines the linearity of the oscillator (1.3.6).

Applying the definitions of variables (I, θ) we can express the old variables (p, q) in terms of the new ones

$$p = p(I, \theta); \qquad q = q(I, \theta). \tag{1.3.9}$$

Due to the cyclicity of the variable θ, i.e. phase shift in θ by $2\pi n$, for an integer n, does not change the expression (1.3.9), q and p can be expanded in a Fourier series:

$$\begin{aligned} q = q(I, \theta) &= \sum_{n=-\infty}^{\infty} a_n(I)\, \mathrm{e}^{\mathrm{i}n\theta}; \\ p = p(I, \theta) &= \sum_{n=-\infty}^{\infty} b_n(I)\, \mathrm{e}^{\mathrm{i}n\theta}, \end{aligned} \tag{1.3.10}$$

where the coefficients a_n and b_n are equal to:

$$\begin{aligned} a_n &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}n\theta} q(I, \theta); \\ b_n &= \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-\mathrm{i}n\theta} p(I, \theta). \end{aligned} \tag{1.3.11}$$

The Fourier harmonics (1.3.11) determine the spectral properties of the system. As the variables (p, q) are real, the following relations for the expansion coefficients can be presented:

$$a_{-n} = a_n^*; \qquad b_{-n} = b_n^*. \tag{1.3.12}$$

Out of general considerations, the important asymptotic property of the system's spectrum can be found [6] if the system motion is periodic:

$$\begin{aligned} a_n &\sim \exp(-n/N_0); \\ b_n &\sim \exp(-n/N_0) \qquad (n \to \infty), \end{aligned} \tag{1.3.13}$$

where N_0 is a constant, defining an effective number of harmonics within the spectrum of the system's oscillations. For $n > N_0$ the amplitudes of the Fourier harmonics are exponentially small. Specifically, in the case of a linear oscillator, the oscillations (1.3.7) contain exactly one harmonic.

1.4 The nonlinear pendulum

The nonlinear pendulum is a common physical model. This is due to the fact that many problems concerning oscillations can be more or less easily reduced to the equations of the nonlinear pendulum. Its Hamiltonian has the form

$$H = \tfrac{1}{2}\dot{x}^2 - \omega_0^2 \cos x, \tag{1.4.1}$$

where unit mass is assumed, i.e. $p = \dot{x}$, and ω_0 is the frequency of weak oscillations. The equation of motion for a nonlinear pendulum is as follows:

$$\ddot{x} + \omega_0^2 \sin x = 0 \tag{1.4.2}$$

and its phase portrait is as shown in Fig. 1.4.1. The singularities are of the elliptic type ($\dot{x} = 0$, $x = 2\pi n$), and saddles ($\dot{x} = 0$, $x = 2\pi(n+1)$); $n = 0, \pm 1, \ldots$. When $H < \omega_0^2$, the trajectories correspond to the pendulum's oscillations (finite motion), in the case of $H > \omega_0^2$, to the pendulum's rotation (infinite motion). Trajectories with $H = \omega_0^2$ are separatrices. The solution on a separatrix can be obtained if we substitute

$$H = H_s = \omega_0^2 \tag{1.4.3}$$

in the equation (1.4.1). This gives us the following equation:

$$\dot{x} = \pm 2\omega_0 \cos(x/2). \tag{1.4.4}$$

Having supplied the initial condition: $t = 0$, $x = 0$, we get the solution in the form:

$$x = 4 \arctan \exp(\pm \omega_0 t) - \pi. \tag{1.4.5}$$

Two whiskers of a separatrix (one entering the saddle and another leaving it) correspond to the different signs of t. With the help of (1.4.4) we can obtain from (1.4.5)

$$v = \dot{x} = \pm \frac{2\omega_0}{\cosh \omega_0 t}. \tag{1.4.6}$$

The solution of this form is called a soliton.