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The ontology of number

The ontogenesis of number

> This day and age we're living in give cause for apprehension,
> With speed and new invention,
> And things like third dimension,
> Yet we get a trifle weary, with Mr. Einstein's theory,
> So we must get down to earth,
> At times relieve the tension.
> No matter what the progress, or what may yet be proved,
> The simple facts of life are such they cannot be removed.
> . . .
> The fundamental things apply,
> As time goes by. *Herman Hupfeld, 1930*

When it comes to mankind's use and understanding of numbers, what are the simple facts of life? Are there any fundamental things which always apply? Are numbers part of a reality which exists independently of the lives and deaths of individual human beings and the rise and fall of civilizations'? (Restivo 1983: 231). If this is so, then the characteristics of numbers are largely defined in terms of the ways in which they can combine with one another, according to the rules of what we call 'mathematics', or, in confining its application to numbers, simply 'arithmetic'. In that case, following Frege, we are confronted with 'the principal problem of arithmetic. [...] How do we apprehend logical objects, in particular numbers?' (Cited Hodes 1984). But whatever may be fundamental about numbers, one must still ask what are the simple facts of life, and much of this book is an attempt to answer this question in its relation to numbers. The foundations cannot be laid without, first, having some sort of *cognition* of numbers, and some command over numerical techniques. Both are determined in different ways by context: a compulsive gambler on cockfights in Bali will derive his cognition of numbers from the general culture of the island, but the application will make use of special skills developed so as to win as much as possible.

The cognitive base, examined in detail in chapter 2, cannot exist without some set of *signs* representing the series of *natural* numbers, 1, 2, 3 ... The representation is inevitably partial, since the series of natural numbers is infinite.[1] It takes the form of words 'one', 'two', 'three' and so on, representing the lowest numbers in the series, up to a limit determined by the resources of the local language. These words, in whatever language it may be, are almost always distinctive (Hurford 1987: 4), and divorced from the rest of the vocabulary. The young child may first become acquainted with these words by means of a ritual *activity* (ibid.: 106) of *symbolic* counting (Lancy 1983: 142), such as is sometimes incorporated into nursery rhymes, but the symbols are *unnatural*. This may be the reason why anthropologists have been guarded in their approach to numerical symbolism. As Lévi-Strauss has pointed out in a recent work (1985), in the societies which traditionally interested anthropologists the symbolism was rooted in natural phenomena, so that – to give but one example – in parts of the world as widely separated as Australasia and North America, populations were divided into clans bearing the names of animals.[2] Such systems point towards the concrete, where unnatural systems can only point towards the abstract. Although at an elementary level the use of numbers may not be mathematical at all (and some cultures may go no further than this), sooner or later 'arithmetic is involved in the interpretation of numeral expressions, though nowhere else in language, so numerals are, prima facie, odd' (Hurford 1987: 5). And since arithmetic has to do with numbers rather than the numerals which denote numbers, one has, in some way, to transcend the symbolic base. Much of this book is taken up with this process in a variety of different contexts. But it is sufficient if 'for the time being, a number is simply something that can be named by a numerical expression' (ibid.: 8).

The logical problem is then to discover what exactly numbers are: this requires relating the *symbolic* numbers, according to the definition just given, to the series of natural numbers, 1, 2, 3,... Since, however, as chapter 3 shows, a viable society need not even have 'words' for a handful of low numbers, let alone rules of syntax for combining them to represent higher numbers, this requirement is by no means easy to satisfy. The point would seem to be reinforced by the fact that the indefinite extension of the realm of numbers to incorporate all the demands of the cognitive domain at some point requires a written notation. This provides then the earliest record of 'cognitive style', that is the primary mode of thinking about numbers (Davis and Hersh 1983: 307), in the historical development of advanced numerical systems. The evidence is unmistakable that the cognitive basis of number, at this point in history, was defined by the need to record quantities of concrete objects (Friberg 1984: 78) and not by that of facilitating the development of arithmetic in the abstract.

One would think, intuitively, that prehistoric cognitive style could not have been less concrete, but this conclusion is contradicted by research in the many pre-literate societies known to modern anthropology. Lévi-Strauss' study of the importance of groups of ten in the mythology of North-American Indians, shows how the purely arithmetical properties of the number, 10, determined both structure and interpretation (Lévi-Strauss 1971). Indeed the invention of the written recording of numbers occurred in a context calculated, if anything, to demythologize numbers.

We do not have to accept 'the dogma that the intuition of the natural number system is universal is not tenable in the light of historical, pedagogical or anthropological experience' (Davis and Hersh 1983) at the cost of being deprived of any firm foundation for continued discussion. If some anthropologists, for example Sahlins, would seem to welcome such deprivation, the purpose of this book, as that of any scientific work, is to present its subject matter to readers who have some cultural identification with its author. I accept the general application of the proposition stated by Brainerd (1973b: 247), that 'the structure of mathematics is in some sense isomorphic with the structure of reality', to all numerical praxis, traditional as well as modern. My conclusion will then be that this proposition also characterises, at least in part, most traditional cognitive styles. This view also makes it possible to define numeracy in terms of the ability to understand and work with the structure of mathematics, at such level of development as is feasible given the linguistic resources of the local culture. More particularly, where the definition is such as to include mathematical notation and calculating instruments, however rudimentary, it becomes clear that the greater these resources are, the more significant is the potential role of numeracy. It is therefore realistic to recognise that levels of numeracy not only vary greatly from one culture to another, but also that they can do so within any one culture, particularly one that is at all complex. This will often prove to be the result of a traditional pre-literate society having to come to terms with an imposed system of school education based on Western models.

There is, none the less, no inherent need for the user to be conscious of any abstract properties of numbers, even at the elementary level of realising that some are odd, while others are even. Numbers as a purely formal system are independent of external reality. Then, even if 'nobody need be aware that there is an isomorphism[3] between the two' (Hofstadter 1980: 53), in practice it is perfectly possible to project properties of the formal system on to a real situation, if the structures of the two are congruent. I provide many such instances in this book.[4] For the moment, however, the point can be made by equating the two sides of a street with the two formal categories of numbers, odd and even, and then to rule that on even days of the month cars may be parked on one side, and on odd days, on the other. If one can imagine a non-numerate society, which still had cars, there would be no great problem in establishing a rule which achieved the same result without appealing to the isomorphism. On the other hand, the problem would be of a quite different order if parking were regulated according to whether or not the number of the day of the month was prime. This would depend not only upon a fairly advanced development of the abstract properties of the formal system, but also upon establishing a systematic relationship between ordinal and cardinal numbers. This, as the following section will show, cannot be taken for granted.

The general truth, however, is that where numerical techniques are known – even at the elementary level of separating numbers into odd and even – they are used as 'paradigms of identity' (Bloor 1983: 92), so that 'mathematics is normative', (ibid.: 91, citing Wittgenstein 1964, V, 41) and numeracy becomes an indispensable cultural accomplishment. The reality is that use is made of certain operations and techniques which are central to the training given to children, and are fastened upon, so as to become memorable patterns: 'it's an ethnological fact – it's something to do with the way we live' (Wittgenstein 1970: 244, 249). The truth of the proposition that a 'language-game, even in mathematics, can implicate our whole life' (Bloor 1983: 100), will be abundantly proved in the following pages,[5] for mathematics is not so much a science, but a 'language for other sciences' (Davis and Hersh 1983: 343). Further, it is 'hard to think of any branch of mathematics which does not depend on real numbers' (ibid.: 370).

The question then is: what are the characteristics of mathematics, in its application to traditional use and understanding of number? The answer is to be found partly in praxis, and partly in cognitive style. As to the former, it is useful to start with Resnick and Ford's (1981) 'partitioning of the "mathematics space" into number facts, algorithms, and problem solving' (Lancy 1983: 187). Number facts are basically what are learnt in mastering the numerical vocabulary of one's own language. Algorithms are more or less standard procedures for applying this knowledge in culturally defined contexts. The constant use of such algorithms may generate, at least among some individuals, true mathematical concepts, which can be recognised because they do not 'embody regularities of our sensory experiences of the physical environment, but regularities of these regularities, and relations between them at a high order of abstraction' (Skemp 1980: 9). This introduces, once again, the fundamental question of the relation of the power of abstraction to the use and understanding of numbers. Here, following Lancy (1983: 110), there must be 'some point in the acquisition and practice of mathematics when understanding number as an abstraction becomes necessary. It is still not clear where that point lies, nor the extent to which growing up in a nonnumerically ordered society, hinders the attainment of this level of abstraction'. On the other hand, in certain numerical contexts, such as music or games, the power of abstraction may be found, even where it is absent in the rest of the local culture.

Where the power of abstraction is insufficiently developed, traditional numeracy is constrained by the fact that '...counting and ordering are the only *experimental* techniques in arithmetic' and that these are insufficient to 'prove even the most elementary theorem, such that the number of primes is unlimited' (Hofstadter 1980: 58). In the result traditional use of numbers tends to be characterised by the endless, and often extremely complex, application of algorithms – that is, of fixed procedures, which produce some new statement out of basic numerical facts. Elementary arithmetic, as taught in primary school, is of this kind, even though the algorithms now taught are hardly a hundred years old (Davis and Hersh 1983: 91). As for cognitive style, the continued use of algorithms means that mathematics is seen, essentially, as a means for generating results. This is just like the modern computer, which of its nature can only apply pre-set procedures. This is quite different from the insights generated by the dialectic of modern mathematics (Davis and Hersh 1983: 182f), which are such that 'today many, perhaps most, mathematicians have no [...] conviction of the objective existence of the objects they study' (ibid.: 252). In traditional numeracy, the whole art lies in interpretation and application, rather in the way that a modern heart specialist looks at an electrocardiogram, without needing to understand much about the actual mechanics of the instrument he is using.[6]

This leads to considerable problems, for it means that the numerical institutions of traditional societies must be functional, which leads to the question as to whether there is a sort of cultural ecology of numbers. The cases I cite in this book will make clear that in some cultures far more use is made of numbers than in others: at one extreme there are, for instance, the Balinese who seem not to be able to do anything without numbers, while at the other, there are the Bemba of Zambia, who would readily dispense with them altogether. The explanation has surprisingly often a historical basis in the process of diffusion: it is no accident that the Balinese are Hindu wet-rice cultivators, so that both their religion, and an economy based upon the meticulous control of irrigation, provide some explanation for their obsession with numbers. Even at this level, numerical institutions can be ordered in terms of their efficiency and user-friendly characteristics: in the realm of language, for instance, the place-value system of written numerals has generally superseded all competing systems. In particular, the more abstract a numerical institution – and such institutions are of their nature abstract – the greater its power of diffusion. Once again, both music and games provide many examples.

It does not follow, however, that the present book requires a categorical answer to the question I ask in the first paragraph, about the independent reality of mathematics. Is it then sufficient to accept the 'constructivist' view of the Dutch mathematician, L. E. J. Brouwer, that in mathematics nothing can be accepted as meaningful, or even be recognised as existing, unless it can be derived by a finite process whose starting point is the natural numbers, although this is hardly acceptable to most modern mathematicians (Davis and Hersh 1983: 334 and Russell 1980)? Or must we accept the logicist view, attributed to Plato, that mathematical objects are real, and their existence an objective fact, quite independent of our knowledge of them?[7] (We can also be content with a subjective variant, attributed to Kant according to which 'the truths of[...]arithmetic are forced upon us by the way our minds work; this explains why they are supposedly true and independent of experience. The intuitions of time and space[...]are objective in the sense that they are universally valid for all human minds' (ibid.: 329).) Finally there is the formalist view, according to which mathematics is a game, with rules made up of symbols and formulae (ibid.: 410), which is certainly the approach of many traditional numerical institutions. In this last case, one has an instance of the contrast (discussed in greater detail in chapter 11) between constitutive systems, where formal rules define the field of operation, and regulative systems which govern operations carried out in a field with its own separate existence.

The real issue is whether the theme of the present book can be treated holistically. The final chapter will reconsider this question in the light of all the preceding instances of the use and understanding of numbers. The question to bear in mind is whether the constructivist, logicist and formalist views are believed, each in their own contexts, because each corresponds to a certain conceptualisation – or application – of numbers: according to the first view, numbers are no more than what man makes of them; according to the second numbers are, in the words of Einstein (cited Davis and Hersh 1983: 68), 'the symbolic counterpart to the universe', an ideal, pre-existing non-temporal reality, open, at least in part, to discovery by man; according to the third, man makes the rules of the game, but the powers then given to numbers are outside his control. In a book in which the main focus is on culture and cognition, the question as to which view is correct, or in what circumstances, can be left open: in every separate case, the facts must speak for themselves. All that is needed is some workable definition of number, combined with a systematic treatment of the different instances of number which can occur in a traditional culture. The following section of this chapter provides the necessary definition, on the basis of the classification into ordinal, natural and cardinal numbers, which proves to be at least compatible with almost any cognitive domain, while in the final section I define the scope of the book, and the way in which its subject matter is organised.

Ordinal, natural and cardinal numbers

Order (according to Russell 1920: 31) is a property of the members of any set in which there is a recognised relation of precedence and succession. Once such a relationship is established between any two members of the set, it is a simple matter to prove that it imposes one single order upon all the members. Any recognisable order, however it occurs, always depends upon the application of some principle: this is a requirement with any collection of objects, however heterogeneous. The contents of a shopping basket, for example, could be ordered on the basis of weight or cost. If no such principle is available, then no order can be imposed, but this case is unimportant, for where order is desired, necessity proves to be the mother of invention when it comes to finding an acceptable principle. The opposite case is much more common: circumstances dictate the principle to be applied, so that, for instance, seniority within the family is determined according to age. Any principle depending on time has a built-in order, since with any two instants of time, one must precede the other.[8] I shall show in chapter 7 how the principle of order determines the traditional use and understanding of time.

The case of time always shows how order can be represented, according to context, by its own terminology. Therefore, we have the distinctive names for the days of the week or the months of the year, a principle which in some traditional societies has been extended to recurring periods of much longer duration.[9] It does not matter that the terms defining order constantly, and even frequently, recur, at least so long as ambiguity is avoided. Indeed the use of such terms can be extremely convenient, as can be seen by considering a sentence such as: 'In Holland grocery stores are closed on Tuesday afternoons.' Of course such a statement, judged cognitively, takes much for granted, but this causes no difficulty in any context in which it is likely to be made. The cognitive demands can be even stricter, as can be seen from the sentence, 'I have been invited to a party next Tuesday afternoon', which is only meaningful if it can be located within one definite week.[10]

The case of cyclic order, implicit in much of the terminology of time, is extremely fundamental, of common occurrence and will constantly recur in this book. It is more difficult to establish in purely logical terms (Russell 1920: 40f.), and the cases vary according to the number of points in the cycle. In the binary case, represented by the Yin–Yang principle discussed in chapter 4, the endless alternation between two points make it impossible to establish precedence. Does night follow day, or day follow night? But with three points the problem can be solved, as I illustrate in chapter 10 with the Japanese game of Janken. With four points, one has the familiar example of the compass points, and with five, the different orders of the five elements, also discussed in chapter 4. These cases of low-number cycles are significant for illustrating the isomorphisms recognised between different instances of cycles with the same number of points, so that, for instance, the Chinese used the five elements to organise space on the basis of the four cardinal points and a centre at which opposing forces cancelled each other, thereby preserving the harmony of the whole system (Needham 1980: 286). But at the same time the mathematical properties of the group constituted by the five elements were independent of this or any other particular application.

The existence of cyclic sets also demonstrates that ordered sets may contain only a finite number of members, but in practical use ordering always implies a closed domain. A phrase such as 'in the two hundred and fiftieth year of his life' plainly does not refer to any ordinary mortal, though logically and grammatically there is nothing wrong with it. The important distinction to be made is between sets, which by definition have a fixed number of ordered members, such as the days of the week, and those for which no precise upper limit need be known. The first case allows for the concept introduced by Gerschel (1962: 696) of the 'nombre marginal', defined as 'un nombre qui n'existe pas, puis qu'il surpasse d'une unité le dernier nombre réel, mais qui, en vertu des lois de la fiction, gagne en extension ce qu'il perd en compréhension'. This would explain the significance of the 1001 nights: Shahrazād was saved because she succeeded in surviving one night longer than the number of nights which could be counted. In this case it is no coincidence that 1000 is the cube of 10, but it is particularly in relation to binary numbers that the *nombre marginal* becomes significant. The case of 33, which exceeds by one the fifth power of 2, is an example which I shall use a number of times in this study.

Returning to the principle of isomorphism, once this is recognised, then the next logical step is to order the isomorphic sets according to the number of members contained in each, so that the first term of the series will designate those with two members, the second those with three, and so on. (The one member set is a quite different construction, left out of the present analysis, largely for historical reasons[11].) The insight, which allows this step to be taken, is the breakthrough to the use of natural numbers, as we know them: 1, 2, 3,.... Up to this point the existence of the series of natural numbers, however implicit in the ordered sets within the cognitive domain, can still be excluded from it. What it does require, on the present analysis, is the capacity to categorise in a perfectly abstract way, so that number provides the means of putting into one and the same category completely heterogeneous collections of things, whose only common property is that they have the same number of members (Russell 1920: 14).

Applying this lesson to traditional societies, one is faced with the problem, noted by Lancy (1983: 66–7), that grouping collections together, on the basis of numerical equivalence, is not necessarily part and parcel of every culture which uses numbers. It is not even certain whether there is an intellectual basis for such a process, ripe for development in every individual, which will actually be realised if the cultural factors are favourable. What Western culture takes for granted, in its command of numbers, may be the result of a historical process, leading to the extreme division of labour, and the general use of writing and formal schooling, a process that is only now just beginning to make an impact on non-Western cultures.

The use of natural numbers (without which there is no question of numeracy) implies a system of symbols apt to designate them. In the first instance these symbols may be taken to be in the form of spoken words, although there may be no logical necessity for this. It would seem to follow that once the use of the natural numbers is admitted, these symbolic numbers, or 'numerals', can be used to 'count' the number of members of any ordered set, so as to establish an 'ordinal number' corresponding to the last member of that set. In this way the numbers *one*, *two*, *three* ... up to *eleven* can be used to count the months from January to November, so as to make November the *eleventh* month. This establishes the ordinal number as the relation number of a well-ordered series. It is thus a species of serial number (Russell 1920: 93). The dilemma according to Russell, is that 'a set of terms has all the orders of which it is capable. Sometimes one order is so much more familiar and natural to our thoughts that we are inclined to regard it as the order of that set of terms;[12] but this is a mistake' (ibid.: 29f), even though 'in counting, it is necessary to take the objects counted in a certain order' (ibid.: 17). His conclusion is supported by experiments carried out in Papua-New Guinea which fail to show any relationship between an understanding of the number concept and the notion of *ordinality* – that things can be arranged in order of magnitude (Lancy 1983: 142). Lancy notes the case of Ponam, one of the countless languages of Papua New Guinea, in which there is no general system of ordinals beyond *first*, *middle* and *last*, even though the numerical system is otherwise relatively advanced.[13] On the other hand, the elementary cognition of a pre-literate population in Papua New-Guinea can hardly be taken as a confirmation, let alone a proof, of advanced and esoteric propositions in mathematical logic (which belong to a conceptual domain quite beyond any such cognition), nor even as sufficient to falsify the results of experimental work in development psychology, such as that done by Brainerd (1973a, 1973b).

Once the number of a set is divorced from the order of its members, the individual identity of each number is lost. This is inherent in the transformation from ordinal to cardinal numbers. An ordinal is an adjective, which is only significant if it qualifies something; a cardinal number is a substantive, which can stand alone. The cardinal number is an abstraction, and as I have already noted, any particular cardinal number is the only common property of all collections which have that number of members. How then do we know how many members there are in any collection of things before us? The commonsense answer (at least in our own culture) is to count them, but, as Russell says, 'counting cannot define numbers, because numbers are used in counting' (Russell 1920).[14] It follows, then, that if we want to use numbers for counting, they must have some definite meaning, and be something more than symbols capable of being manipulated in the processes of arithmetic. Where Russell is content to find this meaning in the logical theory of arithmetic, we are wise not to follow him, for we would then lose all possible contact with the cognitive domain of any traditional culture. It is better simply to accept that counting depends upon an intellectual resource, consisting of a set of symbols, in the first instance taking the form of spoken words – in English starting with 'one, two, three' – whose actual range of application will in practice be much narrower than is theoretically possible. These words will therefore always be an imperfect, because essentially incomplete, representation of cardinal numbers as established and defined by the logical theory of arithmetic. Russell, himself, noted that 'it must have required many ages to discover that a brace of pheasants and a couple of days were both instances of the number 2: the degree of abstraction is far from easy', even though he probably did not realise that there were cultures, such as that of the Hopi of the American desert, in which cognition did not extend this far.

The linguistic resources of any cognitive domain largely determine how cardinal numbers are conceived. When these resources are sufficient, then the cardinal numbers can combine with each other in ways that we recognise as belonging to arithmetic, and in the general context of language this property is quite unique (Hurford 1975: 3). The scope of arithmetic is greatly enhanced by two factors: the first is a good, written notation, the second, the availability of efficient means of calculation. Deficiencies in the first factor can largely be made good by the second, so that the abacus made advanced calculation possible long before the place-value principle on which it was based was incorporated into the written form of numbers. Effective calculation, then, only requires a suitable set of algorithms: in Japan and China these governed the use of the abacus, the written numerals being used only for the purposes of recording numbers.[15]

The deficiencies of notation and the means for calculation has often led to pre-occupation with elementary arithmetical structures which require only the use of low numbers. An instance of this is the magic square – of which an example is given in figure 1 – invented by the Chinese more than two thousand years ago, and brought by them to a remarkable degree of perfection (Needham 1959: 55ff.). The Chinese tradition also shows the importance of alternative numerical systems, such as the binary system which is the basis of the I Ching (which I discuss in chapter 4) and which also provides the winning strategy for the game of *Nim*, described in chapter 10. Indeed, the Chinese use and understanding of binary numbers foreshadowed the principle, established by von Neumann in relation to computers – which also work with binary numbers – that the same codes represent both the numerical input and the calculations imposed upon it by the program. Once again the autonomy of cardinal numbers is established in the context of arithmetic.

To summarise, the scheme set out in this section assumes a sort of numerical ontogenesis based on the order: ordinal numbers,[16] natural numbers, cardinal numbers. This corresponds to the process of psychogenesis established by development psychology (Brainerd 1973b: 221f.) – a subject I