Introduction

The algebraic K-groups $K_*(A)$ and the algebraic L-groups $L_*(A)$ are the obstruction groups to the existence and uniqueness of geometric structures in homotopy theory, via Whitehead torsion and the Wall finiteness and surgery obstructions. In the topological applications the ground ring A is the group ring $\mathbf{Z}[\pi]$ of the fundamental group π. For K-theory a geometric structure is a finite CW complex, while for L-theory it is a compact manifold. The lower K- and L-groups are the obstruction groups to imposing such a geometric structure after stabilization by forming a product with the i-fold torus

$$T^i = S^1 \times S^1 \times \ldots \times S^1 ,$$

arising algebraically as the codimension i summands of the K- and L-groups of the i-fold Laurent polynomial extension of A

$$A[\pi_1(T^i)] = A[z_1,(z_1)^{-1},z_2,(z_2)^{-1},\ldots,z_i,(z_i)^{-1}] .$$

The object of this text is to provide a unified algebraic framework for lower K- and L-theory using chain complexes, leading to new computations in algebra and to further applications in topology.

The 'fundamental theorem of algebraic K-theory' of Bass [7] relates the torsion group K_1 of the Laurent polynomial extension $A[z,z^{-1}]$ of a ring A to the projective class group K_0 of A by a naturally split exact sequence

$$\begin{aligned} 0 \longrightarrow K_1(A) \longrightarrow & K_1(A[z]) \oplus K_1(A[z^{-1}]) \\ & \longrightarrow K_1(A[z,z^{-1}]) \longrightarrow K_0(A) \longrightarrow 0 . \end{aligned}$$

The lower K-groups $K_{-i}(A)$ of [7] were defined inductively for $i \geq 1$ to fit into natural split exact sequences

$$\begin{aligned} 0 \longrightarrow K_{-i+1}(A) \longrightarrow & K_{-i+1}(A[z]) \oplus K_{-i+1}(A[z^{-1}]) \\ & \longrightarrow K_{-i+1}(A[z,z^{-1}]) \longrightarrow K_{-i}(A) \longrightarrow 0 , \end{aligned}$$

generalizing the case $i = 0$.

The quadratic L-groups of polynomial extensions were first studied by Wall [84], Shaneson [72], Novikov [48] and Ranicki [57]. The free quadratic L-groups L^h_* of the Laurent polynomial extension $A[z,z^{-1}]$ ($\bar{z} = z^{-1}$) of a ring with involution A were related in [57] to the projective quadratic L-groups L^p_* of A by natural direct sum decompositions

$$L^h_n(A[z,z^{-1}]) = L^h_n(A) \oplus L^p_{n-1}(A) .$$

The lower quadratic L-groups $L^{\langle -i \rangle}_*(A)$ of [57] were defined inductively for $i \geq 1$ to fit into natural direct sum decompositions

$$L^{\langle -i+1 \rangle}_n(A[z,z^{-1}]) = L^{\langle -i+1 \rangle}_n(A) \oplus L^{\langle -i \rangle}_{n-1}(A)$$

with $L_*^{\langle 0\rangle} \equiv L_*^p$, generalizing the case $i = 0$ with $L_*^{\langle 1\rangle} \equiv L_*^h$.

An algebraic theory unifying the torsion of Whitehead [88], the finiteness obstruction of Wall [83] and the surgery obstruction of Wall [84] was developed in Ranicki [60]–[69] using chain complexes in any additive category $\mathbb{A}$. This approach is used here in the K- and L-theory of polynomial extensions and the lower K- and L-groups. Chain complexes offer the usual advantage of a direct passage from topology to algebra, avoiding preliminary surgery below the middle dimension. A particular feature of the exposition is the insistence on relating the geometric transversality properties of manifolds to the algebraic transversality properties of chain complexes.

The computation $Wh(\mathbb{Z}^i) = 0$ $(i \geq 1)$ of Bass, Heller and Swan [8] gives the lower K- and L-groups of $\mathbb{Z}$, which are used (more or less explicitly) in Novikov's proof of the topological invariance of the rational Pontrjagin classes, the work of Kirby and Siebenmann on high-dimensional topological manifolds, Chapman's proof of the topological invariance of Whitehead torsion and West's proof that compact ANRs have the homotopy type of finite CW complexes. A systematic treatment of homeomorphisms of compact manifolds requires the study of non-compact manifolds, using the controlled algebraic topology of spaces initiated by Chapman, Ferry and Quinn. In this theory topological spaces are equipped with maps to a metric space X, and the notions of maps, homotopy, cell exchange, surgery etc. are required to be small when measured in X. The original simple homotopy theory of Whitehead detects if a PL map is close to being a PL homeomorphism. The controlled simple homotopy theory detects if a continuous map is close to being a homeomorphism, by considering the size of the point inverses.

After an initial lull, the lower K- and L-groups have found many applications in the controlled and bounded topology of non-compact manifolds, stratified spaces and group actions on manifolds. The following alphabetic list of references is representative: Anderson and Hsiang [3], Anderson and Munkholm [4], Anderson and Pedersen [5], Bryant and Pacheco [13], Carlsson [16], Chapman [18], Farrell and Jones [25], Ferry and Pedersen [28], Hambleton and Madsen [30], Hambleton and Pedersen [31], Hughes [35], Hughes and Ranicki [36], Lashof and Rothenberg [42], Madsen and Rothenberg [45], Pedersen [51], Pedersen and Weibel [53], [54], Quinn [56], Ranicki and Yamasaki [70], [71], Siebenmann [75], Svennson [80], Vogell [81], Weinberger [86], Weiss and Williams [87], Yamasaki [89].

Karoubi [38] and Farrell and Wagoner [26] were motivated by the proof of Bott periodicity using operators on Hilbert space and by the simple homotopy theory of infinite complexes (respectively) to describe the lower K-groups of a ring A as the ordinary K-groups of rings of infinite matrices which are locally finite

$$K_{-i}(A) \;=\; K_0(S^iA) \;=\; K_1(S^{i+1}A) \;\; (i \geq 0) \;,$$

with S^iA the i-fold suspension ring: the suspension SA is the ring defined by the quotient of the ring of locally finite countable matrices with entries in A by the ideal of globally finite matrices. Gersten [29] used the ring suspension to define a non-connective spectrum $\mathbf{K}(A)$ with homotopy groups

$$\pi_i(\mathbf{K}(A)) \;=\; K_i(A) \;\; (i \in \mathbb{Z}) \;.$$

The applications of the lower K- and L-groups to manifolds generalize the end invariant of Siebenmann [73], which interprets the Wall finiteness obstruction $[W] \in \widetilde{K}_0(\mathbb{Z}[\pi])$ of an open n-dimensional manifold W with one tame end as the obstruction to closing the end, assuming that $n \geq 6$ and that $\pi = \pi_1(W)$ is also the fundamental group of the end. The following conditions on W are equivalent:

(i) $[W] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi])$,

(ii) W is homotopy equivalent to a finite CW complex,

(iii) W is homeomorphic to the interior of a closed n-dimensional manifold M,

(iv) the cellular chain complex $C(\widetilde{W})$ of the universal cover $\widetilde{W}$ of W is chain equivalent to a finite f.g. free $\mathbb{Z}[\pi]$-module chain complex.

The product $W \times S^1$ has end invariant

$$[W \times S^1] \;=\; 0 \in \widetilde{K}_0(\mathbb{Z}[\pi \times \mathbb{Z}]) \;,$$

so that $W \times S^1$ is homeomorphic to the interior of a closed $(n+1)$-dimensional manifold N. However, if $[W] \neq 0 \in \widetilde{K}_0(\mathbb{Z}[\pi])$ then N is not of the form $M \times S^1$ for a closed n-dimensional manifold M with interior homeomorphic to W.

Motivated by controlled and the closely related bounded topology, Pedersen [49], [50] expressed the lower K-group $K_{-i}(A)$ $(i \geq 0)$ of a ring A both as the class group of the idempotent completion $\mathbb{P}_i(A)$ of the additive category $\mathbb{C}_i(A)$ of $\mathbb{Z}^i$-graded A-modules which are f.g. free in each grading, with bounded morphisms, and as the torsion group of $\mathbb{C}_{i+1}(A)$

$$K_{-i}(A) \;=\; K_0(\mathbb{P}_i(A)) \;=\; K_1(\mathbb{C}_{i+1}(A)) \;\; (i \geq 0) \;.$$

These K-theory identifications are obtained here by direct chain complex constructions, and extended to corresponding identifications of the lower L-groups of a ring with involution A as the L-groups of additive categories with involution

$$L_n^{\langle -i\rangle}(A) = L_{n+i}(\mathbb{P}_i(A)) = L_{n+i+1}(\mathbb{C}_{i+1}(A)) \ \ (i \geq 0) .$$

The method can also be used to express the lower L-groups as the L-groups of multiple suspensions

$$L_n^{\langle -i\rangle}(A) = L^p_{n+i}(S^i A) = L^h_{n+i+1}(S^{i+1}A) \ \ (i \geq 0) .$$

For $i = 0$ this is an unpublished result of Farrell and Wagoner (cf. Wall [84, p.251]).

The *open cone* of a subspace $X \subseteq S^k$ is the metric space

$$O(X) = \{tx \in \mathbb{R}^{k+1} \,|\, t \in [0,\infty), x \in X\} .$$

Open cones are especially important in the topological applications of bounded K- and L-theory, because (roughly speaking) the controlled algebraic topology of X is the bounded algebraic topology of $O(X)$.

Given a filtered additive category $\mathbb{A}$ and a metric space X let $\mathbb{C}_X(\mathbb{A})$ be the filtered additive category of X-graded objects in $\mathbb{A}$ and bounded morphisms defined by Pedersen and Weibel [53], and let $\mathbb{P}_X(\mathbb{A})$ be the idempotent completion of $\mathbb{C}_X(\mathbb{A})$. Let $\mathbb{P}_0(\mathbb{A})$ denote the idempotent completion of $\mathbb{A}$ itself, and let $\mathbf{K}(\mathbb{A})$ be the non-connective algebraic K-theory spectrum of $\mathbb{A}$, with homotopy groups

$$\begin{aligned} \pi_i(\mathbf{K}(\mathbb{A})) &= K_i(\mathbb{P}_0(\mathbb{A})) \ \ (i \in \mathbb{Z}) \\ &= K_i(\mathbb{A}) \ \ (i \neq 0) . \end{aligned}$$

The main result of Pedersen and Weibel [54] shows that the algebraic K-theory assembly map

$$H^{lf}_*(X;\mathbf{K}(\mathbb{A})) \longrightarrow K_*(\mathbb{P}_X(\mathbb{A}))$$

is an isomorphism for $X = O(Y)$ an open cone on a compact polyhedron $Y \subseteq S^k$, so that

$$K_*(\mathbb{P}_{O(Y)}(\mathbb{A})) = H^{lf}_*(O(Y);\mathbf{K}(\mathbb{A})) = \tilde{H}_{*-1}(Y;\mathbf{K}(\mathbb{A})) .$$

In particular, the algebraic K-groups $K_*(\mathbb{P}_{O(Y)}(\mathbb{A}))$ of the open cone of a union $Y = Y^+ \cup Y^- \subseteq S^k$ of compact polyhedra fit into a Mayer-Vietoris exact sequence

$$\begin{aligned} \dots \longrightarrow K_i(\mathbb{P}_{O(Y^+\cap Y^-)}(\mathbb{A})) &\longrightarrow K_i(\mathbb{P}_{O(Y^+)}(\mathbb{A})) \oplus K_i(\mathbb{P}_{O(Y^-)}(\mathbb{A})) \\ \longrightarrow K_i(\mathbb{P}_{O(Y)}(\mathbb{A})) &\xrightarrow{\partial} K_{i-1}(\mathbb{P}_{O(Y^+\cap Y^-)}(\mathbb{A})) \\ \longrightarrow K_{i-1}(\mathbb{P}_{O(Y^+)}(\mathbb{A})) &\oplus K_{i-1}(\mathbb{P}_{O(Y^-)}(\mathbb{A})) \\ &\longrightarrow K_{i-1}(\mathbb{P}_{O(Y)}(\mathbb{A})) \longrightarrow \dots . \end{aligned}$$

Carlsson [16] extended the methods of [54] to metric spaces other than open cones, obtaining a Mayer-Vietoris exact sequence for the algebraic

K-groups $K_*(\mathsf{P}_X(\mathbb{A}))$ of a union $X = X^+ \cup X^-$ of arbitrary metric spaces

$$\begin{aligned} \dots \longrightarrow \varinjlim_b K_i(\mathsf{P}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) \longrightarrow K_i(\mathsf{P}_{X^+}(\mathbb{A})) \oplus K_i(\mathsf{P}_{X^-}(\mathbb{A})) \\ \longrightarrow K_i(\mathsf{P}_X(\mathbb{A})) \xrightarrow{\partial} \varinjlim_b K_{i-1}(\mathsf{P}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) \\ \longrightarrow K_{i-1}(\mathsf{P}_{X^+}(\mathbb{A})) \oplus K_{i-1}(\mathsf{P}_{X^-}(\mathbb{A})) \longrightarrow K_{i-1}(\mathsf{P}_X(\mathbb{A})) \longrightarrow \dots , \end{aligned}$$

with

$$\begin{aligned} \mathcal{N}_b(X^+,X^-,X) &= \mathcal{N}_b(X^+,X) \cap \mathcal{N}_b(X^-,X) \\ &= \{x \in X \,|\, d(x,x^+), d(x,x^-) \le b \text{ for some } x^+ \in X^+,\ x^- \in X^-\} \end{aligned}$$

the intersection of the b-neighbourhoods of X^+ and X^- in X. This exact sequence will be obtained in §4 for $i = 1$ using elementary chain complex methods, with the connecting map

$$\partial \ :\ K_1(\mathsf{C}_X(\mathbb{A})) \longrightarrow \varinjlim_b K_0(\mathsf{P}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A}))$$

defined by sending the torsion $\tau(E)$ of a contractible finite chain complex E in $\mathsf{C}_X(\mathbb{A})$ to the projective class $[E^+]$ of a $\mathsf{C}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})$-finitely dominated subcomplex $E^+ \subseteq E$, corresponding to the end obstruction of Siebenmann [73], [75] of the part of E lying over a b-neighbourhood of X^+ in X. In the special case

$$X = X^+ \cup X^- = \mathbb{R} \ ,\ X^+ = [0,\infty) \ ,\ X^- = (-\infty,0]$$

the algebraic K-groups are such that

$$K_*(\mathsf{C}_{\mathbb{R}^\pm}(\mathbb{A})) = K_*(\mathsf{P}_{\mathbb{R}^\pm}(\mathbb{A})) = 0 \ ,\ K_*(\mathsf{C}_{\mathbb{R}}(\mathbb{A})) = K_*(\mathsf{C}_1(\mathbb{A})) \ .$$

The connecting map in this case is the isomorphism of Pedersen and Weibel [53]

$$\partial \ :\ K_1(\mathsf{C}_1(\mathbb{A})) \cong K_0(\mathsf{P}_0(\mathbb{A})) \ .$$

The algebraic properties of modules and quadratic forms over a Laurent polynomial extension ring $A[z,z^{-1}]$ are best studied using algebraic transversality techniques which mimic the geometric transversality technique for the construction of fundamental domains of infinite cyclic covers of compact manifolds. The linearization trick of Higman [34] was the first such algebraic transversality result, leading to the method of Mayer-Vietoris presentations developed by Waldhausen [80]. In §10 this method is used to obtain a split exact sequence

$$\begin{aligned} 0 \longrightarrow K_1(\mathbb{A}) \longrightarrow K_1(\mathbb{A}[z]) \oplus K_1(\mathbb{A}[z^{-1}]) \\ \longrightarrow K_1(\mathbb{A}[z,z^{-1}]) \longrightarrow K_0(\mathsf{P}_0(\mathbb{A})) \longrightarrow 0 \end{aligned}$$

for any filtered additive category $\mathbb{A}$. The split projection $K_1(\mathbb{A}[z,z^{-1}])$

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$\longrightarrow K_0(\mathbf{P}_0(\mathbb{A}))$ is induced by the embedding of $\mathbb{A}[z,z^{-1}]$ in $\mathbb{C}_1(\mathbb{A})$ as a subcategory with homogenously $\mathbb{Z}$-graded objects. Let $\mathbf{P}_i(\mathbb{A})$ denote the idempotent completion $\mathbf{P}_0(\mathbb{C}_i(\mathbb{A}))$ of the bounded $\mathbb{Z}^i$-graded category $\mathbb{C}_i(\mathbb{A})$. The above sequence is used in §11 to recover the expression due to Pedersen and Weibel [53] of the lower K-groups of $\mathbb{A}$ as

$$K_{-i}(\mathbb{A}) \;=\; K_0(\mathbf{P}_i(\mathbb{A})) \;=\; K_1(\mathbb{C}_{i+1}(\mathbb{A}))\quad (i \geq 1)\ ,$$

using polynomial extensions (in the spirit of Bass [7]) instead of the delooping machinery.

The quadratic L-groups $L_*(\mathbb{A})$ are defined in Ranicki [68] for any additive category $\mathbb{A}$ with an involution, as the cobordism groups of quadratic Poincaré complexes in $\mathbb{A}$. The intermediate quadratic L-groups $L^J_*(\mathbb{A})$ are defined for a $*$-invariant subgroup $J \subseteq K_0(\mathbb{A})$ to be the cobordism groups of quadratic Poincaré complexes in $\mathbb{A}$ with the projective class required to belong to J. The algebraic transversality method is applied in §14 to obtain a Mayer-Vietoris exact sequence for the quadratic L-groups $L_*(\mathbb{C}_X(\mathbb{A}))$ of a union $X = X^+ \cup X^-$

$$\begin{aligned}\dots \longrightarrow \varinjlim_b L^{J_b}_n(\mathbf{P}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) &\longrightarrow L_n(\mathbb{C}_{X^+}(\mathbb{A})) \oplus L_n(\mathbb{C}_{X^-}(\mathbb{A}))\\ \longrightarrow L_n(\mathbb{C}_X(\mathbb{A})) &\xrightarrow{\partial} \varinjlim_b L^{J_b}_{n-1}(\mathbf{P}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) \longrightarrow \dots\ ,\end{aligned}$$

with

$$J_b \;=\; \ker(K_0(\mathbf{P}_{\mathcal{N}_b(X^+,X^-,X)}(\mathbb{A})) \longrightarrow K_0(\mathbf{P}_{X^+}(\mathbb{A})) \oplus K_0(\mathbf{P}_{X^-}(\mathbb{A})))\ .$$

In particular, the quadratic L-groups $L_*(\mathbb{C}_{O(Y)}(\mathbb{A}))$ of the open cone of a union $Y = Y^+ \cup Y^- \subseteq S^k$ fit into a Mayer-Vietoris exact sequence

$$\begin{aligned}\dots \longrightarrow L^J_n(\mathbf{P}_{O(Y^+\cap Y^-)}(\mathbb{A})) &\longrightarrow L_n(\mathbb{C}_{O(Y^+)}(\mathbb{A})) \oplus L_n(\mathbb{C}_{O(Y^-)}(\mathbb{A}))\\ \longrightarrow L_n(\mathbb{C}_{O(Y)}(\mathbb{A})) &\xrightarrow{\partial} L^J_{n-1}(\mathbf{P}_{O(Y^+\cap Y^-)}(\mathbb{A})) \longrightarrow \dots\ ,\end{aligned}$$

with

$$J \;=\; \ker(K_0(\mathbf{P}_{O(Y^+\cap Y^-)}(\mathbb{A})) \longrightarrow K_0(\mathbf{P}_{O(Y^+)}(\mathbb{A})) \oplus K_0(\mathbf{P}_{O(Y^-)}(\mathbb{A})))\ .$$

In the important special case

$$Y^{\pm} \;=\; \{\pm 1\}\ ,\ Y \;=\; Y^+ \cup Y^- \;=\; S^0$$

the open cones are

$$O(Y^{\pm}) \;=\; \mathbb{R}^{\pm}\ ,\ O(Y) \;=\; \mathbb{R}$$

and

$$L_*(\mathbb{C}_{\mathbb{R}^{\pm}}(\mathbb{A})) \;=\; 0\ ,\ L_*(\mathbb{C}_{\mathbb{R}}(\mathbb{A})) \;=\; L_*(\mathbb{C}_1(\mathbb{A}))\ ,\ J = K_0(\mathbf{P}_0(\mathbb{A}))\ ,$$

so that the connecting maps define isomorphisms

$$\partial \;:\; L_*(\mathbb{C}_1(\mathbb{A})) \;\cong\; L_{*-1}(\mathbf{P}_0(\mathbb{A}))\ .$$

In §17 the lower L-groups of a filtered additive category category $\mathbb{A}$ are defined inductively by

$$L_*^{\langle 0\rangle}(\mathbb{A}) = L_*(\mathbb{P}_0(\mathbb{A})) ,$$
$$L_*^{\langle -i\rangle}(\mathbb{A}[z,z^{-1}]) = L_*^{\langle -i\rangle}(\mathbb{A}) \oplus L_{*-1}^{\langle -i-1\rangle}(\mathbb{A}) \quad (i \geq 1) .$$

The isomorphisms ∂ are used to obtain the expressions

$$L_n^{\langle -i\rangle}(\mathbb{A}) = L_{n+i}(\mathbb{P}_i(\mathbb{A})) = L_{n+i+1}(\mathbb{C}_{i+1}(\mathbb{A})) \quad (n \geq 0\,,\, i \geq 1) .$$

The lower L-groups of a ring with involution A are the lower L-groups of the additive category with involution of based f.g. free A-modules. The ultimate lower quadratic L-groups of $\mathbb{A}$ are defined by

$$L_n^{\langle -\infty\rangle}(\mathbb{A}) = \varinjlim_i L_n^{\langle -i\rangle}(\mathbb{A}) \quad (n \in \mathbb{Z}) ,$$

and as in Ranicki [69] there is defined an algebraic L-theory spectrum $\mathbb{L}_.^{\langle -\infty\rangle}(\mathbb{A})$ with homotopy groups

$$\pi_*(\mathbb{L}_.^{\langle -\infty\rangle}(\mathbb{A})) = L_*^{\langle -\infty\rangle}(\mathbb{A}) .$$

The Mayer-Vietoris exact sequence of §14 shows that the algebraic L-theory assembly map of [69, Appendix C]

$$H_*^{lf}(X;\mathbb{L}_.^{\langle -\infty\rangle}(\mathbb{A})) \longrightarrow L_*^{\langle -\infty\rangle}(\mathbb{C}_X(\mathbb{A}))$$

is an isomorphism for $X = O(Y)$ the open cone of a compact polyhedron $Y \subseteq S^k$, so that

$$\begin{aligned} L_*^{\langle -\infty\rangle}(\mathbb{C}_{O(Y)}(\mathbb{A})) &= H_*^{lf}(O(Y);\mathbb{L}_.^{\langle -\infty\rangle}(\mathbb{A})) \\ &= \tilde{H}_{*-1}(Y;\mathbb{L}_.^{\langle -\infty\rangle}(\mathbb{A})) . \end{aligned}$$

In §20 the chain complex methods are used to provide an abstract treatment of the obstruction theory of Farrell [21], [22] and Siebenmann [76] for fibering a manifold over the circle S^1.

In Ranicki and Yamasaki [70] the lower K-theory algebra is applied to the controlled topology of Chapman-Ferry-Quinn, obtaining systematic proofs of the results of Chapman and West on the topological invariance of Whitehead torsion and the homotopy finiteness of compact ANRs. The bounded surgery theory of Ferry and Pedersen [28] is the topological context for which the lower L-theory algebra presented here is most directly suited. However, in Ranicki and Yamasaki [71] the algebra will be applied to the controlled surgery theory of Quinn [56] and Yamasaki [89].

An earlier version of this text was issued as Heft 25 (1990) of the Mathematica Gottingensis. I should like to take this opportunity of thanking the Sonderforschungsbereich ‘Geometrie und Analysis’ in Göttingen for its manifold hospitality on various occasions since its inception in 1984.

§1. Projective class and torsion

This section is a brief recollection from Ranicki [64], [65] of the algebraic theory of finiteness obstruction and torsion in an additive category $\mathbb{A}$.

Give $\mathbb{A}$ the split exact structure: a sequence in $\mathbb{A}$

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{j} C \longrightarrow 0$$

is *exact* if there exists a morphism $k : C \longrightarrow B$ such that

$$\text{(i) } jk = 1 : C \longrightarrow C ,$$

$$\text{(ii) } (i\ k) : A \oplus C \longrightarrow B \text{ is an isomorphism in } \mathbb{A}.$$

The *class group* $K_0(\mathbb{A})$ is the abelian group with one generator $[A]$ for each object A in $\mathbb{A}$, subject to the relations

$$\text{(i) } [A] = [A'] \text{ if } A \text{ is isomorphic to } A',$$

$$\text{(ii) } [A \oplus B] = [A] + [B] \text{ for any objects } A, B \text{ in } \mathbb{A}.$$

A chain complex C in $\mathbb{A}$ is *n-dimensional* if $C_r = 0$ for $r < 0$ and $r > n$

$$C : \ldots \longrightarrow 0 \longrightarrow C_n \xrightarrow{d} C_{n-1} \longrightarrow \ldots \longrightarrow C_1 \xrightarrow{d} C_0 .$$

A chain complex C in $\mathbb{A}$ is *finite* if it is n-dimensional for some $n \geq 0$.

The *class* of a finite chain complex C in $\mathbb{A}$ is the chain homotopy invariant defined by

$$[C] = \sum_{r=0}^{\infty} (-)^r [C_r] \in K_0(\mathbb{A}) .$$

The *k-fold suspension* of a chain complex C is the chain complex $S^k C$ defined for any $k \in \mathbb{Z}$ by a dimension shift $-k$

$$d_{S^k C} = d_C : (S^k C)_r = C_{r-k} \longrightarrow (S^k C)_{r-1} = C_{r-k-1} .$$

If C is n-dimensional and $n+k \geq 0$ the k-fold suspension $S^k C$ is $(n+k)$-dimensional, with class

$$[S^k C] = (-)^k [C] \in K_0(\mathbb{A}) .$$

The *idempotent completion* $\mathbb{P}_0(\mathbb{A})$ of $\mathbb{A}$ is the additive category with objects (A, p) defined by the objects A of $\mathbb{A}$ together with a projection $p = p^2 : A \longrightarrow A$. A morphism $f : (A,p) \longrightarrow (B,q)$ in $\mathbb{P}_0(\mathbb{A})$ is a morphism $f : A \longrightarrow B$ in $\mathbb{A}$ such that $qfp = f : A \longrightarrow B$. The full embedding

$$\mathbb{A} \longrightarrow \mathbb{P}_0(\mathbb{A}) \ ; \ A \longrightarrow (A, 1)$$

will be used to identify $\mathbb{A}$ with a subcategory of $\mathbb{P}_0(\mathbb{A})$. The *reduced class group* of $\mathbb{P}_0(\mathbb{A})$ is defined by

$$\widetilde{K}_0(\mathbb{P}_0(\mathbb{A})) = \text{coker}(K_0(\mathbb{A}) \longrightarrow K_0(\mathbb{P}_0(\mathbb{A}))) .$$

Given a ring A let $\mathbb{B}^f(A)$ be the additive category of based f.g. free A-modules. The idempotent completion $\mathbb{P}_0(\mathbb{B}^f(A))$ is isomorphic to the additive category $\mathbb{P}(A)$ of f.g. projective A-modules, and

$$K_0(\mathbb{P}_0(\mathbb{B}^f(A))) = K_0(\mathbb{P}(A)) = K_0(A) ,$$
$$\widetilde{K}_0(\mathbb{P}_0(\mathbb{B}^f(A))) = \widetilde{K}_0(A) .$$

Let $(\mathbb{B}, \mathbb{A} \subseteq \mathbb{B})$ be a pair of additive categories, with $\mathbb{A}$ full in $\mathbb{B}$. A chain complex in $\mathbb{B}$ is *homotopy $\mathbb{A}$-finite* if it is chain equivalent to a finite chain complex in $\mathbb{A}$. An *$\mathbb{A}$-finite domination* (D, f, g, h) of a chain complex C in $\mathbb{B}$ is a finite chain complex D in $\mathbb{A}$ together with chain maps $f : C \longrightarrow D$, $g : D \longrightarrow C$ and a chain homotopy $h : gf \simeq 1 : C \longrightarrow C$. The *projective class* of an $\mathbb{A}$-finitely dominated chain complex C in $\mathbb{B}$ is the class of any finite chain complex (D, p) in $\mathbb{P}_0(\mathbb{A})$ which is chain equivalent to $(C, 1)$ in $\mathbb{P}_0(\mathbb{B})$

$$[C] = [D, p] \in K_0(\mathbb{P}_0(\mathbb{A})) .$$

See Ranicki [64] for an explicit construction of such a (D, p) from an $\mathbb{A}$-finite domination of C. The reduced projective class is such that $[C] = 0 \in \widetilde{K}_0(\mathbb{P}_0(\mathbb{A}))$ if and only if C is homotopy $\mathbb{A}$-finite.

A *chain homotopy projection* (D, p) is a chain complex D together with a chain map $p : D \longrightarrow D$ such that there exists a chain homotopy

$$p \simeq p^2 : D \longrightarrow D .$$

A *splitting* (C, f, g) of (D, p) is a chain complex C together with chain maps $f : C \longrightarrow D$, $g : D \longrightarrow C$ such that $gf \simeq 1 : C \longrightarrow C$, $fg \simeq p : D \longrightarrow D$. The *projective class* of a chain homotopy projection (D, p) with D a finite chain complex in $\mathbb{A}$ was defined in Lück and Ranicki [44] by

$$[D, p] = [C] \in K_0(\mathbb{P}_0(\mathbb{A}))$$

for any splitting (C, f, g) of (D, p) in $\mathbb{P}_0(\mathbb{A})$. See [44] for an explicit construction of an object (D_ω, p_ω) in $\mathbb{P}_0(\mathbb{A})$ such that

$$[D, p] = [D_\omega, p_\omega] - [D_{odd}] \in K_0(\mathbb{P}_0(\mathbb{A})) ,$$

with

$$D_{even} = D_0 \oplus D_2 \oplus D_4 \oplus \dots ,$$
$$D_{odd} = D_1 \oplus D_3 \oplus D_5 \oplus \dots ,$$
$$D_\omega = D_{even} \oplus D_{odd} = D_0 \oplus D_1 \oplus D_2 \oplus \dots .$$

The reduced projective class is such that $[D, p] = 0 \in \widetilde{K}_0(\mathbb{P}_0(\mathbb{A}))$ if and only if (D, p) has a splitting (C, f, g) with C a finite chain complex in $\mathbb{A}$.

A finite chain complex C in $\mathbb{A}$ is *round* if

$$[C] \;=\; 0 \in K_0(\mathbb{A})\ .$$

The projective class of an $\mathbb{A}$-finitely dominated chain complex C is such that $[C]=0\in K_0(\mathbb{P}_0(\mathbb{A}))$ if and only if C is chain equivalent to a round finite chain complex in $\mathbb{A}$. The reduced projective class is such that $[C]=0\in \widetilde{K}_0(\mathbb{P}_0(\mathbb{A}))$ if and only if C is homotopy $\mathbb{A}$-finite.

PROPOSITION 1.1 *If in an exact sequence of chain complexes in* $\mathbb{B}$

$$0 \longrightarrow C \longrightarrow D \longrightarrow E \longrightarrow 0$$

any two of C, D, E *are* $\mathbb{A}$*-finitely dominated then so is the third, and the projective classes are related by the sum formula*

$$[C]-[D]+[E] \;=\; 0 \in K_0(\mathbb{P}_0(\mathbb{A}))\ .$$

□

The *torsion group* of an additive category $\mathbb{A}$ is the abelian group $K_1(\mathbb{A})$ with one generator $\tau(f)$ for each automorphism $f: M\longrightarrow M$, subject to the relations

(i) $\tau(gf: M\longrightarrow M\longrightarrow M) \;=\; \tau(f:M\longrightarrow M)+\tau(g:M\longrightarrow M)\ ,$

(ii) $\tau(i^{-1}fi: L\longrightarrow M\longrightarrow M\longrightarrow L) \;=\; \tau(f:M\longrightarrow M)\ ,$

(iii) $\tau(f\oplus f': M\oplus M'\longrightarrow M\oplus M')$

$$=\; \tau(f:M\longrightarrow M)+\tau(f':M'\longrightarrow M')\ .$$

A *stable isomorphism* $[f]: L\longrightarrow M$ between objects in $\mathbb{A}$ is an equivalence class of isomorphisms $f: L\oplus X\longrightarrow M\oplus X$ in $\mathbb{A}$, under the equivalence relation

$$(f: L\oplus X\longrightarrow M\oplus X) \;\sim\; (f': L\oplus X'\longrightarrow M\oplus X')$$

if the automorphism

$$\alpha \;=\; (f'^{-1}\oplus 1)(f\oplus 1)\;:\; L\oplus X\oplus X' \longrightarrow L\oplus X\oplus X'$$

has torsion $\tau(\alpha)=0\in K_1(\mathbb{A})$.

The composite of stable isomorphisms

$$[f]\;:\;L\longrightarrow M\ ,\ [g]\;:\;M\longrightarrow N$$

represented by

$$f\;:\;L\oplus X\longrightarrow M\oplus X\ ,\ g\;:\;M\oplus Y\longrightarrow N\oplus Y$$

is the stable isomorphism $[gf]: L\longrightarrow N$ represented by the composite

$$(g\oplus 1)(f\oplus 1)\;:\;L\oplus X\oplus Y\longrightarrow N\oplus X\oplus Y\ .$$

A stable automorphism $[f]: L\longrightarrow L$ has a well-defined torsion

$$\tau([f]) \;=\; \tau(f: L\oplus X\longrightarrow M\oplus X)\in K_1(\mathbb{A})\ ,$$