Introduction

This book is an introduction to the theory of iteration of expanding and non-uniformly expanding holomorphic maps and topics in geometric measure theory of the underlying invariant fractal sets. Probability measures on these sets yield information on Hausdorff and other fractal dimensions and properties. The book starts with a comprehensive chapter on abstract ergodic theory, followed by chapters on uniform distance-expanding maps and *thermodynamical formalism*. This material is applicable in many branches of dynamical systems and related fields, far beyond the applications in this book.

Popular examples of the fractal sets to be investigated are Julia sets for rational functions on the Riemann sphere. The theory, which was initiated by Gaston Julia [1918] and Pierre Fatou [1919–1920], has become very popular since the publication of Benoit Mandelbrot's book [Mandelbrot 1982] with beautiful computer generated illustrations. Top mathematicians have since made spectacular progress in the field over the last 30 years.

Consider, for example, the map $f(z) = z^2$ for complex numbers z. Then the unit circle $S^1 = \{|z| = 1\}$ is f-invariant, $f(S^1) = S^1 = f^{-1}(S^1)$. For $c \approx 0, c \neq 0$ and $f_c(z) = z^2 + c$, there still exists an f_c-invariant set $J(f_c)$ called the *Julia set* of f_c, close to S^1, homeomorphic to S^1 via a homeomorphism h satisfying the equality $f \circ h = h \circ f_c$. However, $J(f_c)$ has a fractal shape. For large c the curve $J(f_c)$ pinches at infinitely many points; it may pinch everywhere to become a dendrite, or even crumble to become a Cantor set.

These sets satisfy two main properties, standard attributes of 'conformal fractal sets':

1. Their fractal dimensions are strictly larger than the topological dimension.
2. They are conformally 'self-similar': that is, arbitrarily small pieces have shapes similar to large pieces *via* conformal mappings, here *via* iteration of f.

To measure fractal sets invariant under holomorphic mappings, one applies probability measures corresponding to equilibria in the thermodynamical formalism. This is a beautiful example of the interlacing of ideas from mathematics and physics.

The following *prototype lemma* [Bowen, 1975, Lemma 1.1], resulting from Jensen's inequality applied to the function logarithm, stems from the thermodynamical formalism.

Lemma. *(Finite Variational Principle) For given real numbers $\phi_1, \ldots, \phi_n$ the quantity*

$$F(p_1, \ldots p_n) = \sum_{i=1}^{n} -p_i \log p_i + \sum_{i=1}^{n} p_i \phi_i$$

has maximum value $P(\phi_1, \ldots \phi_n) = \log \sum_{i=1}^{n} e^{\phi_i}$ as $(p_1, \ldots, p_n)$ ranges over the simplex $\{(p_1, \ldots, p_n) : p_i \geq 0, \sum_{i=1}^{n} p_i = 1\}$ and the maximum is attained only at

$$\hat{p}_j = e^{\phi_j} \Big(\sum_{i=1}^{n} e^{\phi_i} \Big)^{-1}.$$

We can read $\phi_i, p_i, i = 1, \ldots, n$ as a function (*potential*), resp. probability distribution, on the finite space $\{1, \ldots, n\}$. The proof follows from the strict concavity of the logarithm function.

Let us further follow Bowen [1975]. The quantity

$$S = \sum_{i=1}^{n} -p_i \log p_i$$

is called the *entropy* of the distribution $(p_1, \ldots, p_n)$. The maximizing distribution $(\hat{p}_1, .., \hat{p}_n)$ is called the *Gibbs* or *equilibrium state*. In statistical mechanics $\phi_i = -\beta E_i$, where $\beta = 1/kT$, T is the temperature of an external 'heat source' and k is a physical (Boltzmann) constant. The quantity $E = \sum_{i=1}^{n} p_i E_i$ is the average energy. The Gibbs distribution thus maximizes the expression

$$S - \beta E = S - \frac{1}{kT} E$$

or, equivalently, minimizes the so-called *free energy* $E - kTS$. Nature prefers states with low energy and high entropy. It minimizes free energy.

The idea of the Gibbs distribution as a limit of distributions on finite spaces of configurations of states (spins, for example) of interacting particles over increasing to infinite, bounded parts of the lattice $\mathbb{Z}^d$ was first introduced in statistical mechanics by Bogolyubov and Hacet [1949] where it plays a fundamental role. It was applied in dynamical systems to study Anosov flows and hyperbolic diffeomorphisms at the end of the 1960s by Ja. Sinai, D. Ruelle and R. Bowen. For more historical remarks see [Ruelle 1978a] or [Sinai 1982]. This theory met the notion of entropy S, borrowed from information theory and introduced by Kolmogorov as an invariant of a measure-theoretic dynamical system.

Later, the usefulness of these notions to the geometric dimensions became apparent. It was already present in [Billingsley 1965], but papers by Bowen [1979] and McCluskey & Manning [1983] were also crucial.

In order to illustrate the idea, consider the following example. Let $T_i : I \to I$, $i = 1, \ldots, n > 1$, where $I = [0, 1]$ is the unit interval, $T_i(x) = \lambda_i x + a_i$, where λ_i, a_i are real numbers chosen in such a way that all the sets $T_i(I)$ are pairwise disjoint and contained in I. Define the limit set Λ as follows:

$$\Lambda = \bigcap_{k=0}^{\infty} \bigcup_{(i_0, \ldots, i_k)} T_{i_0} \circ \cdots \circ T_{i_k}(I) = \bigcup_{(i_0, i_1 \ldots)} \lim_{k \to \infty} T_{i_0} \circ \cdots \circ T_{i_k}(x),$$

the latter union taken over all infinite sequences $(i_0, i_1, \dots)$, the former over sequences of length $k+1$. By our assumptions $|\lambda_j| < 1$: hence the limit exists, and does not depend on x.

It occurs that its Hausdorff dimension is equal to the only number α for which

$$|\lambda_1|^\alpha + \cdots + |\lambda_n|^\alpha = 1.$$

Λ is a Cantor set. It is self-similar with small pieces similar to large pieces with the use of linear (more precisely, affine) maps $(T_{i_0} \circ \cdots \circ T_{i_k})^{-1}$. We call such a Cantor set *linear*. We can distribute a measure μ by setting $\mu(T_{i_0} \circ \cdots \circ T_{i_k}(I)) = \big(\lambda_{i_0} \dots \lambda_{i_k}\big)^\alpha$. Then for each interval $J \subset I$ centred at a point of Λ, its diameter raised to the power α is comparable to its measure μ (this is immediate for the intervals $T_{i_0} \circ \cdots \circ T_{i_k}(I)$). (A measure with this property for all small balls centred at a compact set, in a Euclidean space of any dimension, is called a *geometric measure*.) Hence $\sum(\operatorname{diam} J)^\alpha$ is bounded away from 0 and ∞ for all economical (of multiplicity not exceeding 2) covers of Λ by intervals J.

Note that for each k the measure μ restricted to the space of unions of $T_{i_0} \circ \cdots \circ T_{i_k}(I)$, each such interval viewed as one point, is the Gibbs distribution, where we set $\phi((i_0, \dots, i_k)) = \phi_\alpha((i_0, \dots, i_k)) = \sum_{l=0,\dots,k} \alpha \log \lambda_{i_l}$. The number α is the unique zero of the *pressure function* $\mathrm{P}(\alpha) = \frac{1}{k+1} \log \sum_{(i_0,\dots,i_k)} e^{\phi_\alpha((i_0,\dots,i_k))}$. In this special affine example this is independent of k. In the general non-linear case to define pressure one considers the limit as k goes to ∞.

The family T_i and compositions is an example, very popular in recent years, of *Iterated Function Systems* [Barnsley 1988]. Note that on a neighbourhood of each $T_i(I)$ we can consider $\hat{T} := T_i^{-1}$. Then Λ is an invariant repeller for the distance-expanding map $\hat{T}$.

The relations between dynamics, dimension and geometric measure theory start in our book with the theorem that the Hausdorff dimension of an expanding repeller is the unique zero of the adequate pressure function for sets built with the help of $C^{1+\varepsilon}$ usually non-linear maps in $\mathbb{R}$ or conformal maps in the complex plane $\mathbb{C}$ (or in $\mathbb{R}^d, d > 2$; in this case conformal maps must be Möbius, i.e. a composition of inversions and symmetries, by Liouville's theorem).

This theory was developed for non-uniformly hyperbolic maps or flows in the setting of smooth ergodic theory: see [Katok & Hasselblatt 1995], [Mañé 1987]. Let us also mention [Ledrappier & Young 1985]. See [Pesin 1997] for recent developments. The advanced chapters of our book are devoted to this theory, but we restrict ourselves to complex dimension 1. So the maps are non-uniformly expanding, and the main technical difficulties are caused by critical points, where we have strong contraction, since the derivative by definition is equal to 0 at critical points.

A direction not developed in this book is conformal iterated function systems with infinitely many generators T_i. They occur naturally as return maps in many important constructions, for example for rational maps with parabolic periodic points, or in the *induced expansion* construction for polynomials [Graczyk & Świątek 1998]. See also the recent [Przytycki & Rivera-Letelier 2007]. Beautiful examples are provided by infinitely generated Kleinian groups. For a measure-theoretic background see [Young 1999].

The systematic treatment of iterated function systems with infinitely many generators can be found in [Mauldin & Urbanski 1996] and [Mauldin & Urbański 2003], for example. Recently this has been rigorously explored in the iteration of entire and meromorphic functions.

Below is a short description of the content of the book.

Chapter 1 contains some introductory definitions and basic examples. It is a continuation of this Introduction.

Chapter 2 is an introduction to abstract ergodic theory: here T is a probability measure-preserving transformation. The reader will find proofs of the fundamental theorems: the Birkhoff Ergodic Theorem and the Shannon–McMillan–Breiman Theorem. We introduce entropy and measurable partitions, and discuss canonical systems of conditional measures in Lebesgue spaces, the notion of *natural extension* (inverse limit in the appropriate category). We follow here Rokhlin's Theory [Rokhlin 1949], [Rokhlin 1967]: see also [Kornfeld, Fomin & Sinai 1982]. Next, to prepare for applications for finite-to-one rational maps, we sketch Rokhlin's theory on countable-to-one endomorphisms, and introduce the notion of the Jacobian: see also [Parry 1969]. Finally we discuss mixing properties (K-property, exactness, Bernoulli) and probability laws: the Central Limit Theorem (abbr. CLT), the Law of Iterated Logarithm (LIL), the Almost Sure Invariance Principle (ASIP) for the sequence of functions (random variables on our probability space) $\phi \circ T^n, n = 0, 1, \dots$.

Chapter 3 is devoted to ergodic theory and thermodynamical formalism for general continuous maps on compact metric spaces. The main point here is the so called Variational Principle for pressure: compare with the Finite Variational Principle lemma, above. We also apply functional analysis in order to explain the Legendre transform duality between entropy and pressure. We follow here [Israel 1979] and [Ruelle 1978a]. This material is applicable in *large deviations* and *multifractal analysis*, and is directly related to the uniqueness question of Gibbs states.

In Chapters 2 and 3 we often follow the beautiful book by Peter Walters [Walters 1982].

In Chapter 4 *distance-expanding maps* are introduced. Analogously to Axiom A diffeomorphisms [Smale 1967], [Bowen 1975] or endomorphisms [Przytycki 1976] and [Przytycki 1977], we outline a topological theory: spectral decomposition, specification, Markov partition, and start a 'bounded distortion' play with Hölder continuous functions.

In Chapter 5 thermodynamical formalism and mixing properties of Gibbs measures for open distance-expanding maps T and Hölder continuous potentials ϕ are studied. To a large extent we follow [Bowen 1975] and [Ruelle 1978a]. We prove the existence of Gibbs probability measures (states): m with Jacobian being $\exp(-\phi)$ up to a constant factor, and T-invariant $\mu = \mu_\phi$ equivalent to m. The idea is to use the *transfer operator* $\mathcal{L}_\phi(u)(x) = \sum_{y \in T^{-1}(x)} u(y) \exp \phi(y)$ on the Banach space of Hölder continuous functions u. We prove the exponential convergence $\xi^{-n}\mathcal{L}_\phi^n(u) \to (\int u\,dm)u_\phi$, where ξ is the eigenvalue with the largest absolute value and u_ϕ the corresponding eigenfunction. One obtains $u_\phi = dm/d\mu$. We deduce CLT, LIL and ASIP, and the Bernoulli property for the natural extension.

We provide three different proofs of the uniqueness of the invariant Gibbs measure. The first, and simplest, follows [Keller 1998], the second relies on the Finite Variational Principle, and the third on the differentiability of the pressure function in adequate function directions.

Finally we prove Ruelle's formula:

$$d^2P(\phi + tu + sv)/dt\,ds|_{t=s=0}$$
$$= \lim_{n\to\infty} \frac{1}{n} \int \left(\sum_{i=0}^{n-1} (u \circ T^i - \int u\,d\mu_\phi) \right) \cdot \left(\sum_{i=0}^{n-1} (v \circ T^i - \int v\,d\mu_\phi) \right) d\mu_\phi.$$

This expression for $u = v$ is equal to σ^2 in CLT for the sequence $u \circ T^n$ and measure μ_ϕ.

(In the book we use the letter T to denote a measure-preserving transformation. Maps preserving an additional structure, continuous, smooth or holomorphic for example, are usually denoted by f or g.)

In Chapter 6 (Section 6.1) a metric space with the action of a distance-expanding map f is embedded in a smooth manifold, and it is assumed that the map extends smoothly (or only continuously) to a neighbourhood. Similarly with hyperbolic sets [Katok & Hasselblatt 1995] we discuss basic properties. The intrinsic property of f being an open map on X occurs equivalent to X being repeller for the extension.

We call a repeller X with smoothly extended dynamics a *Smooth Expanding Repeller* (SER).

If an extension is conformal, we say (X, f) is a *conformal expanding repeller* (CER). In Section 6.2 we discuss some distortion theorems and holomorphic motion to be used later in Section 6.4, and in Chapter 9 to prove the analytic dependence of 'pressure' and the Hausdorff dimension of CER on a parameter.

In Section 6.3 we prove that for CER the density $u_\phi = dm/d\mu$ for measures of harmonic potential is real-analytic (and extends so on a neighbourhood of X). This will be used in Chapter 9 for the potential being $-\log|f'|$, in which case μ is equivalent to a Hausdorff measure in the maximal dimension (geometric measure).

In Chapter 7 we provide in detail D. Sullivan's theory classifying $C^{r+\varepsilon}$ line Cantor sets via a *scaling function*, sketched in [Sullivan 1988], and discuss the realization problem [Przytycki & Tangerman 1996]. We also discuss applications for Cantor-like closures of postcritical sets for infinitely renormalizable *Feigenbaum* quadratic-like maps of interval. The infinitesimal geometry of these sets occurs independent of the map, which is one of the famous Coullet–Tresser–Feigenbaum universalities.

In Chapter 8 we provide definitions of various 'fractal dimensions': Hausdorff, box and packing. We also consider Hausdorff measures with gauge functions different from t^α. We prove the 'Volume Lemma' linking, roughly speaking, (global) dimension with local dimensions.

In Chapter 9 we develop the theory of conformal expanding repellers, and relate pressure to the Hausdorff dimension.

Section 9.2 provides a brief exposition of multifractal analysis of the Gibbs measure μ of a Hölder potential on CER X. We rely mainly on [Pesin 1997]. In particular, we discuss the function $F_\mu(\alpha) := \mathrm{HD}(X_\mu(\alpha))$, where $X_\mu(\alpha) := \{x \in X : d(x) = \alpha\}$ and $d(x) := \lim_{r\to 0} \log \mu(B(x,r))/\log r$. The decomposition $X = \bigcup_\alpha (X_\mu(\alpha)) \cup \hat{X}$, where the limit $d(x)$, called the local dimension, does not exist for $x \in \hat{X}$, is called the local dimension spectrum decomposition.

Next we follow the easy (uniform) part of [Przytycki, Urbański & Zdunik 1989] and [Przytycki, Urbański & Zdunik 1991]. We prove that for CER (X, f) and Hölder continuous $\phi : X \to R$, for $\kappa = \mathrm{HD}(\mu_\phi)$, the Hausdorff dimension of the Gibbs measure μ_ϕ (infimum of Hausdorff dimensions of sets of full measure), either $\mathrm{HD}(X) = \kappa$ the measure μ_ϕ is equivalent to Λ_κ, the Hausdorff measure in dimension κ, and is a *geometric measure*, or μ_ϕ is singular with respect to Λ_κ and the right gauge function for the Hausdorff measure to be compared to μ_ϕ is $\Phi(\kappa) = t^\kappa \exp(c\sqrt{\log 1/t \log\log\log 1/t})$. In the proof we use LIL. This theorem is used to prove a dichotomy for the harmonic measure on a Jordan curve ∂, bounding a domain Ω, which is a repeller for a conformal expanding map. Either ∂ is real-analytic, or the harmonic measure is comparable to the Hausdorff measure with gauge function $\Phi(1)$. This yields information about the lower and upper growth rates of $|R'(r\zeta)|$, for $r \nearrow 1$, for almost every ζ with $|\zeta| = 1$ and univalent function R from the unit disc $|z| < 1$ to Ω. This is a dynamical counterpart of Makarov's theory of boundary behaviour for general simply connected domains [Makarov 1985].

We prove, in particular, that for $f_c(z) = z^2 + c, \quad c \neq 0, \quad c \approx 0$ it holds that $1 < \mathrm{HD}(J(f_c)) < 2$.

We show how to express another interesting function in the language of pressure: $\int_{|\zeta|=1} |R'(r\zeta)|^t \, |d\zeta|$ for $r \nearrow 1$.

Finally, we apply our theory to the boundary of the von Koch 'snowflake' and more general Carleson fractals.

Chapter 10 is devoted to Sullivan's rigidity theorem, saying that if two non-linear expanding repellers $(X, f), (Y, g)$ are Lipschitz conjugate (or more generally if there exists a measurable conjugacy that transforms a geometric measure on X to a geometric measure on Y), then the conjugacy extends to a conformal one. This means that measures classify non-linear conformal repellers. This fact, announced in [Sullivan 1986] with only a sketch of the proof, is proved here rigorously for the first time.

(This chapter is one of the oldest chapters in this book; we already made it available in 1991 and many papers have since followed.)

In Chapter 11 we start to deal with non-uniform expanding phenomena. At the heart of this chapter is the proof of the formula $\mathrm{HD}(\mu) = \mathrm{h}_\mu(f)/\chi_\mu(f)$ for an arbitrary f-invariant ergodic measure μ of positive Laypunov exponent $\chi_\mu := \int \log|f'|\, d\mu$.

(The phrase 'non-uniform expanding' is used just to say that we consider (typical points of) an ergodic measure with positive Lyapunov exponent. In higher dimensions one uses the name 'non-uniform hyperbolic' for measures with all Lyapunov exponents non-zero.)

It is so roughly because a small disc around z, whose n-th image is large, has diameter of order $|(f^n)'(z)|^{-1} \approx \exp(-n\chi_\mu)$ and measure $\exp(-n\,\mathrm{h}_\mu(f))$ (the Shannon–McMillan–Breiman theorem is involved here).

Chapter 12 is devoted to conformal measures: that is, probability measures with Jacobian $\mathrm{Const}\exp(-\phi)$ or more specifically $|f'|^\alpha$ in a non-uniformly expanding situation, in particular for any rational mapping f on its Julia set J. It is proved that there exists a minimal exponent $\delta(f)$ for which such a measure exists, and that $\delta(f)$ is equal to each of the following quantities:

Dynamical dimension $\mathrm{DD}(J) := \sup\{\mathrm{HD}(\mu)\}$, where μ ranges over all ergodic f-invariant measures on J of positive Lyapunov exponent.

Hyperbolic dimension $\mathrm{HyD}(J) := \sup\{\mathrm{HD}(Y)\}$, where Y ranges over all Conformal Expanding Repellers in J, or CERs that are Cantor sets.

It is an open problem whether for every rational mapping $\mathrm{HyD}(J) = \mathrm{HD}(J) =$ the box dimension of J, but for many non-uniformly expanding mappings these equalities hold. It is often easier to study the continuity of $\delta(f)$ with respect to a parameter, than study the Hausdorff dimension directly. So one obtains information about the continuity of dimensions due to the above equalities.

Section 12.5 presents a recent approach via pressure for the potential function $-t\log|f'|$, yielding a simple proof of the equalities of the above dimensions, see [Przytycki, Rivera-Letelier & Smirnov 2004].

A large part of this book was written in the years 1990–1992, and was lectured to graduate students by each of us in Warsaw, Yale and Denton. We neglected to finish writing, but recently the methods in Chapter 12, relating hyperbolic dimension to minimal exponent of conformal measure, were unexpectedly used to study the dependence on ε of the dimension of the Julia set for $z^2 + 1/4 + \varepsilon$, for $\varepsilon \to 0$ and other parabolic bifurcations, by A. Douady, P. Sentenac and M. Zinsmeister [1997] and by C. McMullen [1996]. So we decided to make final efforts. Meanwhile good books have appeared on some topics of our book: let us mention [Falconer 1997], [Zinsmeister 1996], [Boyarsky & Góra 1997], [Pesin 1997], [Keller 1998], [Baladi 2000] but a lot of important material in our book is new or has been made more easily accessible.

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1

Basic examples and definitions

Let us start with definitions of dimensions. We shall come back to them in a more systematic way in Chapter 8.

Definition 1.1. Let (X,ρ) be a metric space. We denote by the *upper (lower) box dimension* of X the quantity

$$\overline{\mathrm{BD}}(X)\ (\text{or } \underline{\mathrm{BD}}(X))\ := \limsup(\liminf)_{r\to 0}\frac{\log N(r)}{-\log r},$$

where $N(r)$ is the minimal number of balls of radius r that cover X.

Sometimes the names *capacity* or *Minkowski dimension* or *box-counting dimension* are used. The name 'box dimension' comes from the situation where X is a subset of a Euclidean space $\mathbb{R}^d$. Then one can consider only $r = 2^{-n}$, and $N(2^{-n})$ can be replaced by the number of dyadic boxes $[\frac{k_1}{2^{-n}}, \frac{k_1+1}{2^{-n}}] \times \cdots \times [\frac{k_d}{2^{-n}}, \frac{k_d+1}{2^{-n}}], k_j \in \mathbb{Z}$ intersecting X.

If $\overline{\mathrm{BD}}(X) = \underline{\mathrm{BD}}(X)$ we call the quantity the *box dimension* and denote it by $\overline{\mathrm{BD}}(X)$.

Definition 1.2. Let (X,ρ) be a metric space. For every $\kappa > 0$ we define $\Lambda_\kappa(X) = \lim_{\delta\to 0}\inf\{\sum_{i=1}^\infty(\operatorname{diam} U_i)^\kappa\}$, where the infimum is taken over all countable covers $(U_i, i = 1,2,\dots)$ of X by sets of diameter not exceeding δ. $\Lambda_\kappa(Y)$ defined as above on all subsets $Y \subset X$ is called the κ-th outer *Hausdorff measure*.

It is easy to see that there exists $\kappa_0 : 0 \le \kappa_0 \le \infty$ such that for all $\kappa : 0 \le \kappa < \kappa_0$ $\Lambda_\kappa(X) = \infty$ and for all $\kappa : \kappa_0 < \kappa$ $\Lambda_\kappa(X) = 0$. The number κ_0 is called the *Hausdorff dimension* of X.

Note that if in this definition we replace the assumption: sets of diameter not exceeding δ by equal δ, and $\lim_{\delta\to 0}$ by $\liminf$ or $\limsup$, we obtain the box dimension.

A standard example to compare the two notions is the set $\{1/n, n = 1, 2, \dots\}$ in $\mathbb{R}$. Its box dimension is equal to $1/2$, and the Hausdorff dimension is 0. If one considers $\{2^{-n}\}$ instead one obtains both dimensions as 0. Also, linear Cantor sets, as introduced in the Introduction, have their Hausdorff and box dimensions equal. The reason for this is self-similarity.

Example 1.3. Shift spaces. For every natural number d consider the space Σ^d of all infinite sequences $(i_0, i_1, \dots)$ with $i_n \in \{1, 2, \dots, d\}$. Consider the metric

$$\rho((i_0, i_1, \dots), (i'_0, i'_1, \dots)) = \sum_{n=0}^{\infty} \lambda^n |i_n - i'_n|$$

for an arbitrary $0 < \lambda < 1$. Sometimes it is more convenient to use the metric

$$\rho((i_0, i_1, \dots), (i'_0, i'_1, \dots)) = \lambda^{-\min\{n: i_n \neq i'_n\}},$$

equivalent to the previous one. Consider $\sigma : \Sigma^d \to \Sigma^d$ defined by $\sigma((i_0, i_1, \dots) = (i_1, \dots)$. The metric space (Σ^d, ρ) is called the *one-sided shift space* and the map σ the *left shift*. Often, if we do not specify metric but are interested only in the Cartesian product topology in $\Sigma^d = \{1, \dots, d\}^{\mathbb{Z}^+}$, we use the name *topological shift space*.

One can consider the space $\tilde{\Sigma}^d$ of all two sides infinite sequences $(\dots, i_{-1}, i_0, i_1, \dots)$. This is called the *two-sided shift space*.

Each point $(i_0, i_1, \dots) \in \Sigma^d$ determines its forward trajectory under σ, but is equipped with a Cantor set of backward trajectories. Together with the topology determined by the metric $\sum_{n=-\infty}^{\infty} \lambda^{|n|} |i_n - i'_n|$ the set $\tilde{\Sigma}^d$ can be identified with the *inverse limit* (in the topological category) of the system $\cdots \to \Sigma^d \to \Sigma^d$ where all the maps $\to$ are σ.

Note that the limit Cantor set Λ in the Introduction, with all $\lambda_i = \lambda$, is Lipschitz homeomorphic to Σ^d, with the homeomorphism h mapping $(i_0, i_1, \dots)$ to $\bigcap_k T_{i_0} \circ \cdots \circ T_{i_k}(I)$. Note that for each $x \in \Lambda$, $h^{-1}(x)$ is the sequence of integers $(i_0, i_1, \dots)$ such that for each k, $\hat{T}^k(x) \in T_{i_k}(I)$. This is called a *coding sequence*. If we allow the end points of $T_i(I)$ to overlap, and in particular $\lambda = 1/d$ and $a_i = (i-1)/d$, then $\Lambda = I$ and $h^{-1}(x) = \sum_{k=0}^{\infty} (i_k - 1) d^{-k-1}$.

One generalizes the one (or two) -sided shift space, sometimes called the *full shift space*, by considering the set Σ_A for an arbitrary $d \times d$ matrix $A = (a_{ij}$ with $a_{ij} = 0$ or 1 defined by

$$\Sigma_A = \{(i_0, i_1, \dots) \in \Sigma^d : a_{i_t i_{t+1}} = 1 \quad \text{for every } t = 0, 1, \dots\}.$$

By the definition $\sigma(\Sigma_A) \subset \Sigma_A$. Σ_A with the mapping σ is called a *topological Markov chain*. Here the word *topological* is substantial; otherwise it is customary to think of a finite number of states stochastic process – see Example 1.9.

Example 1.4. Adding machine. A complementary dynamics on Σ^d above is given by the map $T((i_0, i_1, \dots)) = (1, 1, \dots, 1, i_k + 1, i_k + 1, \dots)$, where k is the least integer for which $i_k < d$. Finally $(d, d, d, \dots) + 1 = (1, 1, 1, \dots)$. (This is of course compatible with standard adding, except that here the sequences are infinite to the right and the digits run from 1 to d, rather than from 0 to $d-1$.) Notice that unlike the previous example, with an abundance of periodic trajectories, here each T-trajectory is dense in Σ^d (such a dynamical system is called *minimal*).

Example 1.5. Iteration of rational maps. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a holomorphic mapping of the Riemann sphere $\overline{\mathbb{C}}$. Then it must be rational, i.e. the ratio of two polynomials. We assume that the topological degree of f is at least 2. The *Julia set* $J(f)$ is defined as follows:

$J(f) = \{z \in \overline{\mathbb{C}} : \forall U \ni z, U$ open, the family of iterates $f^n = f \circ \cdots \circ f|_U$, n times, for $n = 1, 2, \ldots$ is not *normal* in the sense of Montel $\}$.

A family of holomorphic functions $f_t : U \to \overline{\mathbb{C}}$ is called *normal* (in the sense of Montel) if it is pre-compact: that is, from every sequence of functions belonging to the family one can choose a sub-sequence uniformly convergent (in the spherical metric on the Riemann sphere $\overline{\mathbb{C}}$) on all compact subsets of U.

$z \in J(f)$ implies for example, that for every $U \ni z$ the family $f^n(U)$ covers all $\overline{\mathbb{C}}$ but at most two points. Otherwise by Montel's theorem $\{f^n\}$ would be normal on U.

Another characterization of $J(f)$ is that $J(f)$ is the closure of repelling periodic points, namely those points $z \in \overline{\mathbb{C}}$ for which there exists an integer n such that $f^n(z) = z$ and $|(f^n)'(z)| > 1$.

There are only a finite number of attracting periodic points, $|(f^n)'(z)| < 1$: they lie outside $J(f)$, which is an uncountable 'chaotic, expansive (repelling)' Julia set. The lack of symmetry between attracting and repelling phenomena is caused by the non-invertibility of f.

It is easy to prove that $J(f)$ is compact, completely invariant: $f(J(f)) = J(f) = f^{-1}(J(f))$, either nowhere dense or equal to the whole sphere (to prove this use Montel's theorem).

For polynomials, the set of points whose images under iterates $f^n, n = 1, 2, \ldots$, tend to ∞, *basin of attraction to* ∞, is connected and completely invariant. Its boundary is the Julia set.

Check that all these general definitions and statements are compatible with the discussion of $f(z) = f_c(z) = z^2 + c$ in the Introduction. As an introduction to this theory we recommend, for example, the books [Beardon 1991], [Carleson & Gamelin 1993], [Milnor 1999] and [Steinmetz 1993].

Figures 1.1–1.3 are computer pictures exhibiting some Julia sets: rabbit, basilica[1] and Sierpiński's carpet of their mating (see [Bielefeld 1990]).

A Julia set can have Hausdorff dimension arbitrarily close to 0 (but not 0) and arbitrarily close to 2 or even exactly 2 (but not the whole sphere). More precisely: a Julia set is always closed and either the whole sphere or nowhere is dense. Recently examples have been found of quadratic polynomials f_c with a Julia set of positive Lebesgue measure (with c in the cardioid; Example 6.1.10): see [Buff & Cheritat 2008]. See also http://picard.ups-tlse.fr/adrien2008/Slides/Cheritat.pdf

[1]The name was proposed by Benoit Mandelbrot [Mandelbrot 1982], impressed by the Basilica San Marco in Venice plus its reflection in flooded Piazza.