

1 Classical fluids

1.1 The distribution function

Macroscopic equations of fluids can be obtained either from macroscopic principles or from the Boltzmann microscopic approach. The first method is more direct and intuitive, but when the mathematical difficulties of the second were eventually overcome, this acquired a higher theoretical interest, and even predicted unknown effects. For astrophysical applications the microscopic approach is preferable, because we do not possess an intuitive perception of many astrophysical fluid systems, with such extreme values of thermodynamic parameters that are clearly beyond human concepts of orders of magnitude. The microscopic approach has provided a powerful tool with which to study many different cosmic problems using a unique and systematic technique.

As is usual, let us introduce the one-particle probability distribution function $f(\vec{r}, \vec{p}, t)$ (where \vec{r} is the position vector, \vec{p} the particle's momentum and t the time), which when multiplied by the six-dimensional phase space volume element $d\tau = d\tau_r d\tau_p = dx dy dz dp_x dp_y dp_z$ gives the total number of particles contained in this volume element.

Our objective is the determination of f . Once f is determined, any macroscopic quantity or property of a fluid system will be easy to obtain.

1.1.1 Number density and mean quantities

Firstly, a one-component fluid, with all molecules having the same mass and properties, will be considered. The number density of particles per spatial volume element can be obtained by

$$n = \int_p f d\tau_p \quad (1.1)$$

where

$$\int_p \equiv \int_{p_x=-\infty}^{\infty} \int_{p_y=-\infty}^{\infty} \int_{p_z=-\infty}^{\infty} \quad (1.2)$$

The number density has an obvious macroscopic importance. If m is the mass of a particle, the density is

$$\rho = mn \quad (1.3)$$

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Let G be a quantity defined for each particle, function of the mass, momentum, and energy. The mean value of G is defined by

$$\langle G \rangle = \frac{1}{n} \int_p G f d\tau_p \quad (1.4)$$

where $\langle G \rangle$, n , and any other macroscopic quantities are functions of \vec{r} and t , but not of \vec{p} . A particular function of obvious macroscopic interest is the mean velocity; taking $G \equiv \vec{v}$:

$$\vec{v}_0 = \langle \vec{v} \rangle = \frac{1}{n} \int_p \vec{v} f d\tau_p \quad (1.5)$$

where \vec{v} is the velocity of a particle, equal to $\frac{\vec{p}}{m}$.

The peculiar velocity of a particle is defined as

$$\vec{V} = \vec{v} - \vec{v}_0 \quad (1.6)$$

so that the velocity of a particle \vec{v} can be decomposed into a mean velocity \vec{v}_0 , identical for all particles in the volume element, plus a peculiar velocity \vec{V} characterizing the random thermal motion of each particle. It is obvious that $\langle \vec{V} \rangle = 0$, and this can easily be confirmed:

$$\langle \vec{V} \rangle = \frac{1}{n} \int_p (\vec{v} - \vec{v}_0) f d\tau_p = \frac{1}{n} \int_p \vec{v} f d\tau_p - \frac{1}{n} \int_p \vec{v}_0 f d\tau_p = \vec{v}_0 - \frac{1}{n} \vec{v}_0 n = 0 \quad (1.7)$$

The temperature is also of obvious macroscopic interest. Though the thermodynamic definition of temperature is very restrictive and in particular requires thermodynamic equilibrium, a kinetic temperature, representative of the magnitude of thermal motions, is introduced here:

$$\frac{3}{2} kT = \frac{1}{2} m \langle V^2 \rangle \quad (1.8)$$

where k is Boltzmann's constant. This definition is only valid for monatomic gases. If the gas is diatomic, $\frac{3}{2}$ must be replaced by $\frac{5}{2}$, and so on. This definition of temperature permits us to use this concept in a system which is not in thermodynamic equilibrium.

1.1.2 Transport fluxes

Let us assume a surface element $d\vec{S}$ inside the fluid, through which molecules freely flow. The number of molecules having a given \vec{v} passing through $d\vec{S}$ is calculated by $f d\tau_r d\tau_p$, where $d\tau_r = (\vec{v} dt) \cdot d\vec{S}$. Therefore $f \vec{v} \cdot d\vec{S} dt d\tau_p$ molecules with momentum between \vec{p} and $\vec{p} + d\vec{p}$ will cross the surface element in dt . Let G again be a mechanical quantity defined for each molecule. Then $G f \vec{v} \cdot d\vec{S} dt d\tau_p$ is the magnitude of G transported through $d\vec{S}$, with a given momentum, in dt . The magnitude of G transported through $d\vec{S}$ in dt by any particle, with any \vec{p} , is

$$\int_p G f \vec{v} \cdot d\vec{S} dt d\tau_p = n \langle G \vec{v} \rangle \cdot d\vec{S} dt \quad (1.9)$$

and per surface and time element $n\langle G\vec{v} \rangle \cdot \vec{u}$, where \vec{u} is the unitary vector with direction $d\vec{S}$. This is the projection of the vector $n\langle G\vec{v} \rangle$ along \vec{u} . This vector is called the transport flux $\vec{\phi}$ of G :

$$\vec{\phi}(G) = n\langle G\vec{v} \rangle \tag{1.10}$$

and has the property that when projected along any direction, it yields the magnitude of G transported per unit area per unit time in this direction.

Let us give some examples of transport fluxes of quantities that can be transported. If G is the mass of a particle, its flux is the vector

$$\vec{\phi} = n\langle m\vec{v} \rangle = mn\langle \vec{v} \rangle = \rho\vec{v}_0 \tag{1.11}$$

Even if there is no net mass flux, G can be transported:

$$\vec{\phi}(G) = n\langle G\vec{v} \rangle = n\langle G(\vec{v}_0 + \vec{V}) \rangle = n\vec{v}_0\langle G \rangle + n\langle G\vec{V} \rangle \tag{1.12}$$

If $\vec{\phi} = 0$, then $\vec{v}_0 = 0$ and the first term is zero, but the second one, $n\langle G\vec{V} \rangle$, may be non-vanishing.

$\vec{\phi}$ has been decomposed into two fluxes: a flux of the fluid carrying the mean value of G and another flux of G which is present even if the fluid is at rest.

As a second important example let G be the momentum. When G is a scalar, $\vec{\phi}$ is a vector. When G is a vector, the flux becomes a second-order tensor. The flux of \vec{p} is

$$\mathcal{R} = n\langle \vec{p}\vec{v} \rangle = \rho\langle \vec{v}\vec{v} \rangle \tag{1.13}$$

Products such as $\vec{p}\vec{v}$ or $\vec{v}\vec{v}$ (without a point) are diadic or external products. Second-order tensors will in general be represented by script capital letters. We now use the relation $\vec{v} = \vec{v}_0 + \vec{V}$ again:

$$\begin{aligned} \mathcal{R} &= \rho\langle (\vec{v}_0 + \vec{V})(\vec{v}_0 + \vec{V}) \rangle \\ &= \rho\langle \vec{v}_0\vec{v}_0 \rangle + \rho\langle \vec{v}_0\vec{V} \rangle + \rho\langle \vec{V}\vec{v}_0 \rangle + \rho\langle \vec{V}\vec{V} \rangle \\ &= \rho\vec{v}_0\vec{v}_0 + \rho\langle \vec{V}\vec{V} \rangle \end{aligned} \tag{1.14}$$

since \vec{v}_0 is a macroscopic quantity, constant in the volume element and can be taken outside the brackets, and $\langle \vec{V} \rangle = 0$. The first term contains $\vec{\phi} = \rho\vec{v}_0$, so that it represents the momentum transport associated with mass flow. The second one represents an internal momentum transport that is present even if the fluid is at rest. This important tensor is called the pressure tensor:

$$\mathcal{P} = \rho\langle \vec{V}\vec{V} \rangle \tag{1.15}$$

which is, by definition, a symmetric second-order tensor.

To enhance familiarity with the pressure tensor, suppose that the fluid is in thermodynamic equilibrium. Then no privileged direction exists; so we can write (1.15) in the form

$$\mathcal{P} = p\delta \tag{1.16}$$

where \mathcal{P} is triply degenerate, δ is the Kroenecker tensor, and p the eigenvalue; p is called the hydrostatic pressure. All off-diagonal components of \mathcal{P} are zero because for any molecule with V_1V_2 a molecule with $-V_1V_2$ can always be found, so the mean of V_1V_2 is zero.

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The use of ‘hydrostatic pressure’, a well-known concept, must be justified. We will do this using two particularly simple examples.

First, suppose that a fluid is contained in a vessel. The walls are pushed perpendicularly with a force per surface element of the wall, given by the momentum which would have been transported through it in dt if the wall had been suppressed, that is, $\mathcal{P} \cdot \vec{u}$, with \vec{u} perpendicular to the wall. The component of this force along \vec{u} , perpendicular to the surface is $(\mathcal{P} \cdot \vec{u}) \cdot \vec{u}$. But because of the threefold degeneracy of \mathcal{P} in equilibrium we have

$$(\mathcal{P} \cdot \vec{u}) \cdot \vec{u} = (\rho \delta \cdot \vec{u}) \cdot \vec{u} = \rho \vec{u} \cdot \vec{u} = p \tag{1.17}$$

Hence, under the equilibrium condition, p is the force per surface element perpendicular to the wall.

Second, assume an ideal monatomic gas in equilibrium. The trace of \mathcal{P} will be $\mathcal{P}_{ii} = 3p = \rho \langle V^2 \rangle$ and, recalling the definition (1.8) of the temperature of a monatomic gas,

$$p = nkT \tag{1.18}$$

which is the equation of state of an ideal gas.

Therefore, at least in equilibrium, the trace of the pressure tensor is three times the hydrostatic pressure. In general, this is adopted as the definition of the hydrostatic pressure.

Suppose, now, that $G \equiv \frac{1}{2}mv^2$, the kinetic energy of a molecule. If we restrict ourselves to a monatomic gas we can obtain the energy flux

$$\begin{aligned} \vec{\Phi} &= n \left\langle \frac{1}{2}mv^2 \vec{v} \right\rangle = \frac{1}{2} \rho \langle v^2 \vec{v} \rangle \\ &= \frac{1}{2} \rho \langle (\vec{v}_0 + \vec{V}) \cdot (\vec{v}_0 + \vec{V})(\vec{v}_0 + \vec{V}) \rangle \\ &= \frac{1}{2} \rho v_0^2 \vec{v}_0 + \frac{1}{2} \rho \vec{v}_0 \cdot \langle \vec{V} \vec{V} \rangle \\ &\quad + \frac{1}{2} \rho \langle \vec{V} \cdot \vec{v}_0 \vec{V} \rangle + \frac{1}{2} \rho \langle V^2 \rangle \vec{v}_0 + \frac{1}{2} \rho \langle V^2 \vec{V} \rangle \\ &= \frac{1}{2} \rho v_0^2 \vec{v}_0 + \mathcal{P} \cdot \vec{v}_0 + \left(\frac{3}{2}kT \right) n \vec{v}_0 + \vec{q} \end{aligned} \tag{1.19}$$

where we have used some tensor properties ($(\vec{A}\vec{B}) \cdot \vec{C} = \vec{A}(\vec{B} \cdot \vec{C}) = \vec{A}\vec{B} \cdot \vec{C}$ for any three vectors; the symmetry of \mathcal{P} has also been used). The vector \vec{q} , the conduction flux, is defined as

$$\vec{q} = \frac{1}{2} \rho \langle V^2 \vec{V} \rangle \tag{1.20}$$

which is again present even if the fluid is at rest. The first term is the macroscopic kinetic energy transported by the fluid. The third term is microscopic kinetic energy transported by the fluid. $\mathcal{P} \cdot \vec{v}_0$ is an interesting term which will be discussed later.

1.1.3 Multicomponent fluids

In a fluid with more than one component, the i th component will have its own distribution function f_i , so that $f_i d\tau_r d\tau_p$ will give the number of molecules of type i contained in the volume element $d\tau_r d\tau_p$ at \vec{r} and \vec{p} in six-dimensional phase space. Then

$$f = \sum_i f_i \tag{1.21}$$

There is a number density n_i for each type of molecule:

$$n_i = \int_p f_i d\tau_p \tag{1.22}$$

and

$$n = \sum_i n_i \tag{1.23}$$

and a partial density

$$\rho_i = m_i n_i \tag{1.24}$$

The density of the mixture is

$$\rho = \sum_i \rho_i \tag{1.25}$$

The equivalent mass of the mixture is defined as

$$m = \frac{\rho}{n} \tag{1.26}$$

which is a weighted average of m_i taking n_i as the weighting factor. For different reasons, not all mean quantities are defined with the same weight for the calculation of the average. The mean velocity \vec{v}_0 of the mixture is defined with the partial mass density as the weighting factor. If $\vec{v}_i = \frac{\vec{p}_i}{m_i}$,

$$\vec{v}_0 = \frac{\sum_i m_i n_i \langle \vec{v}_i \rangle}{\sum_i m_i n_i} \tag{1.27}$$

where, obviously,

$$\langle \vec{v}_i \rangle = \frac{1}{n_i} \int_p f_i \vec{v}_i d\tau_p \tag{1.28}$$

is the mean velocity of the i th component. The peculiar velocity of a particle is again defined as

$$\vec{V}_i = \vec{v}_i - \vec{v}_0 \tag{1.29}$$

but now the mean value of \vec{V}_i for the i th component may be non-vanishing, even where the mean value for the fluid as a whole is zero:

$$\langle \vec{V}_i \rangle = \frac{1}{n_i} \int_p f_i \vec{V}_i d\tau_p = \langle \vec{v}_i \rangle - \vec{v}_0 \tag{1.30}$$

This quantity is macroscopically interesting and is called the diffusion velocity of the i th component. The different diffusion velocities must compensate one another; to be precise:

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$$\begin{aligned} \sum_i m_i n_i \langle \vec{V}_i \rangle &= \sum_i m_i n_i \langle \vec{v}_i - \vec{v}_0 \rangle \\ &= \sum_i m_i n_i \langle \vec{v}_i \rangle - \left(\sum_i m_i n_i \right) \vec{v}_0 \\ &= \left(\sum_i m_i n_i \right) \vec{v}_0 - \left(\sum_i m_i n_i \right) \vec{v}_0 = 0 \end{aligned} \tag{1.31}$$

There is a physical reason for defining \vec{v}_0 as in (1.27) instead of adopting n_i as a weighting factor. If we used n_i we would deduce $\vec{v}_0 = 0$, when the fluid is in fact flowing. Suppose, for instance, a massive component such as uranium is moving to the right with the same speed as a second lighter component such as hydrogen is moving to the left, the numbers of particles of both components being equal. If the weighting factor were n_i we would infer that the fluid was at rest, which would be true for numbers of particles, but not for the mass of the fluid as a whole.

The microscopic velocity \vec{v}_i of a particle can now be written as

$$\vec{v}_i = \vec{v}_0 + \langle \vec{V}_i \rangle + \vec{V}'_i \tag{1.32}$$

\vec{v}_0 and $\langle \vec{V}_i \rangle$ are macroscopic. \vec{V}'_i now represents the individual chaotic thermal velocities. It can easily be found that $\langle \vec{V}'_i \rangle = 0$ for each component. In general, $\langle \vec{V}'_i \rangle$ is much lower than typical values of \vec{V}' and will be considered negligible when compared with the chaotic velocities.

The pressure tensor of an individual component is

$$\mathcal{P}_i = \rho_i \langle \vec{V}'_i \vec{V}'_i \rangle \tag{1.33}$$

To define the temperature of the mixture, an average with the number density as a weighting factor is needed. For monatomic gases:

$$\frac{3}{2} k T_i = \frac{1}{2} m_i \langle V_i'^2 \rangle \tag{1.34}$$

and

$$T = \frac{1}{n} \sum_i n_i T_i \tag{1.35}$$

The physical reason for defining T as in (1.35) is that, under equilibrium conditions, all molecules, irrespective of their mass, have the same energy.

The total flux of the mixture is defined as the sum of the i th fluxes. In particular,

$$\vec{\varphi} = \sum_i \vec{\varphi}_i = \sum_i n_i (\vec{v}_0 + \langle \vec{V}_i \rangle) \tag{1.36}$$

$$\mathcal{P} = \sum_i \mathcal{P}_i = \sum_i \rho_i \langle \vec{V}'_i \vec{V}'_i \rangle \tag{1.37}$$

$$p = \sum_i p_i = \frac{1}{3} \sum_i \sum_j \mathcal{P}_{ij} \tag{1.38}$$

$$\vec{q} = \sum_i \vec{q}_i = \frac{1}{2} \sum_i \rho_i \langle V_i'^2 \vec{V}'_i \rangle \tag{1.39}$$

where the definitions of $\vec{\varphi}_i$, \mathcal{P}_i , p_i , \vec{q}_i are obvious generalizations.

1.1.4 Boltzmann's equation

The objective now is to deduce the differential equations for f_i and integrate them. Once each f_i is known, all functions of macroscopic interest may be calculated from their defining integrals. In particular, we will be able to calculate n_i , n , \bar{v}_0 , $\langle \bar{V}_i \rangle$, T_i , and T .

Let us first assume a one-component fluid. Let \vec{F} be the force acting on a particle, excluding the short-range particle–particle interaction force in a collision. To clarify this concept, we can take a self-gravitating fluid. \vec{F} will be the gravitational force arising from the whole ensemble of particles, but not the gravitational force induced on a particle by a single neighbour particle. \vec{F} can be an external force or it can be produced by the fluid itself.

Particles in the six-dimensional phase space cell $[\vec{r}, \vec{p}]$ (i.e. having their position vector \vec{r} and their momentum \vec{p} in a given volume element $d\tau_r, d\tau_p$) would travel after dt to the cell $[\vec{r} + \vec{v} dt, \vec{p} + \vec{F} dt]$ if there were no collisions. The number of particles in each cell, in the absence of collisions, would then be invariant:

$$f(\vec{r} + \vec{v} dt, \vec{p} + \vec{F} dt, t + dt) d\tau_r d\tau_p = f(\vec{r}, \vec{v}, t) d\tau_r d\tau_p \quad (1.40)$$

Using a series expansion on the left-hand side:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \vec{F} \cdot \nabla_p f = 0 \quad (1.41)$$

where the momentum-gradient ∇_p of f is the vector $\frac{\partial f}{\partial p_i}$. Some collisions will have the effect of propelling particles into the cell $[\vec{r} + \vec{v} dt, \vec{p} + \vec{F} dt]$ which were not previously contained in $[\vec{r}, \vec{p}]$. Others will have the effect of removing particles that were in $[\vec{r}, \vec{p}]$, preventing them from reaching $[\vec{r} + \vec{v} dt, \vec{p} + \vec{F} dt]$. Equation (1.41) must be modified by a term that takes into account the effect of collisions. This term is conventionally called Γ and the result is Boltzmann's equation:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f + \vec{F} \cdot \nabla_p f = \Gamma \quad (1.42)$$

For a multicomponent fluid we have

$$\frac{\partial f_i}{\partial t} + \vec{v}_i \cdot \nabla f_i + \vec{F}_i \cdot \nabla_p f_i = \Gamma_i \quad (1.43)$$

This equation was first derived by Boltzmann in 1872, but the integration was not performed until 1916 by Chapman and Enskog (independently), and even then not for the most general case (only for a dilute monatomic gas not very far from thermodynamic equilibrium). In the absence of collisions and under specified conditions, it was later integrated by Tonks and Langmuir in 1929, Chandrasekhar in 1942 for the fluid comprising galactic stars, Landau in 1946, and others.

Of course, in order to integrate Boltzmann's equation, we should write Γ explicitly as a function of our independent variables \vec{r} , \vec{p} , and t , and of the distribution function f . However, no attempt to describe the integration of this equation in detail will be made here, as it would take up too much space. The reader is referred to classical texts such as those by Chapman and Cowling (1970) or Hirschfelder, Curtiss and Bird (1954). Instead, macroscopic equations consequences of Boltzmann's equation will be derived,

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which do not require an explicit knowledge of the collision term. These macroscopic equations are the fluid dynamic equations. They provide less information than (1.42), as they do not provide $\langle \vec{V}_i \rangle$, \mathcal{P} , and \vec{q} , and to overcome this problem and complete the set of differential equations, some indirect arguments will be invoked in each case, bearing in mind that a formal integration underpins our conclusions.

1.2 **Macroscopic implications**

In order to obtain macroscopic equations in which the collision term is not present, the following procedure may be adopted:

$$\sum_i \int_p (\text{Boltzmann's equation})_i G_i d\tau_p \tag{1.44}$$

Now G_i is a function defined for each particle which has the property of collisional invariance, such as mass, momentum, energy, and any combination of these quantities. Then the collision term gives

$$\sum_i \int_p \Gamma_i G_i d\tau_p = 0 \tag{1.45}$$

because it represents the change in G in a given time element due to collisions between molecules in a space volume element, with any momentum and belonging to any constituent. It is necessary to perform the operation \sum_i because the momentum (for instance) gained by one type of particle will be lost by another type.

In this way, three macroscopic equations can be obtained, corresponding to the conservation of mass, momentum, and energy.

1.2.1 **Continuity equations**

Let us consider mass as the collisional invariant. In this exceptional case, the final sum over all constituents in (1.44) is not necessary because not only is the mass of the whole mixture conserved, but the mass of each individual constituent must also be conserved.

From the first term of Boltzmann's equation, applying (1.44),

$$\int_p \frac{\partial f_i}{\partial t} m_i d\tau_p = m_i \frac{\partial}{\partial t} \int_p f_i d\tau_p = m_i \frac{\partial n_i}{\partial t} \tag{1.46}$$

Note that in this six-dimensional phase space t , \vec{r} , and \vec{p} are independent variables and can have any value assigned by the observer. (For instance, $\vec{p} = m_i \frac{d\vec{r}}{dt}$ cannot be written; \vec{r} and t are independent; and $\frac{d\vec{r}}{dt}$ would be zero.) Due to this fact $\frac{\partial}{\partial t}$ is taken outside the integral. The second term in Boltzmann's equation gives

$$\int_p m_i \vec{v}_i \cdot \nabla f_i d\tau_p = m_i \nabla \cdot \int_p f_i \vec{v}_i d\tau_p = m_i \nabla \cdot (n_i \langle \vec{v}_i \rangle) \tag{1.47}$$

and the third term gives

$$\begin{aligned} \int_p m_i \vec{F}_i \cdot \nabla_p f_i d\tau_p &= m_i F_{i1} \int_{p_1} \int_{p_2} \int_{p_3} \frac{\partial f_i}{\partial p_1} dp_1 dp_2 dp_3 + \dots \\ &= m_i F_{i1} \int_{p_2} \int_{p_3} [f_i]_{-\infty}^{\infty} dp_2 dp_3 + \dots = 0 \end{aligned} \tag{1.48}$$

because $f_i(\infty)$, as well as $f_i(-\infty)$, must vanish, as in any distribution function; otherwise n would be infinite. Another assumption used in obtaining (1.48) is that \vec{F}_i is assumed to be independent of \vec{p} (i.e. independent of \vec{v}_i), as is true for most types of force. This is not always the case and the most important exception is the Lorentz force. This force acts on charged particles; electromagnetic forces and plasmas are considered later in Chapter 4. \vec{F}_i will now be considered independently of \vec{p} and the results obtained will be revised when this is required.

Finally, we obtain

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \langle \vec{v}_i \rangle) = 0 \tag{1.49}$$

which form the continuity equations for each constituent i . They are usually written as

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i (\vec{v}_0 + \langle \vec{V}_i \rangle)) = 0 \tag{1.50}$$

A continuity equation for the mixture is obtained when these equations are multiplied by m_i and added together, taking (1.31) into account:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}_0) = 0 \tag{1.51}$$

Note that \vec{r} and t are independent variables. We are normally interested in knowing the density at a position chosen by us, and at a time chosen by us; there is no sense in calculating $\frac{d\rho}{dt}$. The meaning of $\frac{\partial \rho}{\partial t}$ is obviously the time variation observed in the density, at a fixed point, without any spatial displacement. If the total derivative $\frac{d\rho}{dt}$ were calculated, as $\frac{dx_i}{dt} = 0$ we would obtain $\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t}$. However, there is another time derivative, called the convective derivative, which can be denoted by $\frac{d\rho}{dt}$ without any risk of confusion. In general, $d\rho/dt$ will be given by

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \frac{dx}{dt} + \frac{\partial \rho}{\partial y} \frac{dy}{dt} + \frac{\partial \rho}{\partial z} \frac{dz}{dt} \tag{1.52}$$

Now, we accept that there will be spatial displacements in dt , and that these will actually be taking place macroscopically in the fluid, that is, we accept $\frac{dx}{dt} = v_{0x}$, and so on. Then we write, as a definition of the new symbol $\frac{d}{dt}$,

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v}_0 \cdot \nabla \rho \tag{1.53}$$

To calculate $\frac{d\rho}{dt}$, we follow the fluid motion $\vec{r} = \vec{v}_0 t$: therefore $\frac{d}{dt}$ is the time derivative that would be measured by an observer travelling with the fluid. This observer, who has a velocity $\vec{v}_0(\vec{r}, t)$ at each point, will be called the ‘wet’ observer, in contrast with the inertial observer – the ‘dry’ observer – who measures time variations at a fixed point.

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Equation (1.53) is valid not only for the density, but also for any other quantity, so that the relation between operators can be written as

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \vec{v}_0 \cdot \nabla \tag{1.54}$$

It can also be applied to vectors, and even to \vec{v}_0 itself:

$$\frac{d\vec{v}_0}{dt} = \frac{\partial\vec{v}_0}{\partial t} + \vec{v}_0 \cdot \nabla\vec{v}_0 \tag{1.55}$$

The use of the dry observer’s derivative is preferable for integrating the equations, but the calculation of $\frac{d}{dt}$, the wet observer’s time derivative, or convective derivative, sometimes provides interesting interpretative insights. This is true, for example, for the Milky Way star fluid or for the Universe as a whole, in which we are wet observers.

Another possible expression of the continuity equation might be

$$\frac{d\rho}{dt} + \rho \nabla \cdot \vec{v}_0 = 0 \tag{1.56}$$

which is in fact of less practical interest than (1.51).

The continuity equation is the macroscopic, fluid dynamic form of mass conservation.

1.2.2 The equation of motion

Now $G_i \equiv \vec{p}$, and the operations implied in (1.44) will be carried out. However, \sum_i will be delayed until the end of this section (1.67). Let us define

$$\vec{A}_i = \int_p \Gamma_i \vec{p} d\tau_p \tag{1.57}$$

We know, of course, that

$$\sum_i \vec{A}_i = 0 \tag{1.58}$$

From the first term of Boltzmann’s equation we obtain

$$\int_p \vec{p} \frac{\partial f_i}{\partial t} d\tau_p = m_i \frac{\partial}{\partial t} (n_i (\vec{v}_0 + \langle \vec{V}_i \rangle)) \tag{1.59}$$

From the second term we obtain

$$\begin{aligned} \int_p \vec{p} (\vec{v}_i \cdot \nabla f_i) d\tau_p &= m_i \int_p \vec{v}_i (\vec{v}_i \cdot \nabla f_i) d\tau_p = \nabla \cdot (\rho_i \langle \vec{v}_i \vec{v}_i \rangle) \\ &= \nabla \cdot (\rho_i \langle (\vec{v}_0 + \langle \vec{V}_i \rangle + \vec{V}'_i) (\vec{v}_0 + \langle \vec{V}_i \rangle + \vec{V}'_i) \rangle) \\ &\approx \nabla \cdot (\mathcal{P}_i + \rho_i \vec{v}_0 \vec{v}_0) \end{aligned} \tag{1.60}$$

neglecting $\langle \vec{V}'_i \rangle$ compared to the mean values of \vec{V}'_i , and noting that $\langle \vec{V}'_i \rangle = 0$. Remember that the divergence of a second-order tensor τ is the vector $(\nabla \cdot \tau)_i = \frac{\partial}{\partial x_j} (\tau_{ji})$, that is, the contraction of its gradient.

It will be assumed not only that f_i goes to zero at infinity, but that the distribution moments $[f_i p^\alpha]_{-\infty}^{\infty} = 0$ for any value of α (1, 2, 3, . . .). We will now therefore assume