1

B-splines

1.1 PIECEWISE POLYNOMIAL FUNCTIONS

1.1.1 Notation Let $\mathcal{P}_k$ denote the vector space (over the field $\mathbf{R}$) of polynomials of degree $\leq k$ in one variable. $\mathcal{P}_k$ is then of dimension $k+1$ over $\mathbf{R}$.

Let also be given an interval $[a,b] \subset \mathbf{R}$, an integer $\ell \geq 1$, a set τ of $\ell - 1$ points (τ_i) $(1 \leq i \leq \ell - 1)$ in $]a,b[$

$$a < \tau_1 < \cdots < \tau_{\ell-1} < b$$

called *knots*, and $\ell - 1$ integers r_i such that $0 \leq r_i \leq k$. We will set $a = \tau_o$, $b = \tau_\ell$.

$\mathcal{P}_{k,\tau,r}$ will denote the vector space of the piecewise polynomial functions of degree $\leq k$ on $[a,b]$, with $\mathcal{C}^{r_i - 1}$ continuity at τ_i $(1 \leq i \leq \ell - 1)$ (this means that the function has $r_i - 1$ continuous derivatives at τ_i; for $r_i = 0$ there is no condition).

1.1.2 Lemma

$$dim \mathcal{P}_{k,\tau,r} = (k+1)\ell - \sum_{i=1}^{\ell-1} r_i$$

Proof The space of functions, whose restriction to each of the ℓ intervals $[\tau_i, \tau_{i+1}]$ is a polynomial of degree k, has dimension $(k+1)\ell$.

A basis for this space is given by the functions f_{ij} such that $f_{ij} = (X-\tau_i)^j$ for $X \in [\tau_i, \tau_{i+1}]$ and $f_{ij} = 0$ for $x \notin [\tau_i, \tau_{i+1}]$ $(0 \le i \le \ell - 1, \quad 0 \le j \le k)$.

Let $f \in \mathcal{P}_{k,\tau,r}$, f be equal to the polynomial P_i on $[\tau_i, \tau_{i+1}]$, with $P_i = \sum_{j=0}^k a_j^i (X-\tau_i)^j$; the condition for f to be $\mathcal{C}^{r_i-1}$ at τ_i $(1 \le i \le \ell-1)$ introduces r_i linear relations between the a_i^j's:

$$\begin{cases} a_o^i &= \phi_o(a_o^{i-1}, \ldots, a_k^{i-1}) \\ \vdots & \\ a_{r_i-1}^i &= \phi_{r_i-1}(a_o^{i-1}, \ldots, a_k^{i-1}). \end{cases}$$

These equations are clearly independent, as each one introduces a new variable.

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1.1.3 A basis for $\mathcal{P}_{k,\tau,r}$

Let $(X-\tau_i)_+$ be the function equal to $(X-\tau_i)$ for $X \ge \tau_i$ and 0 for $X \le 0$; one then has $(X-\tau_i)_+ = \mathrm{Sup}(X-\tau_i, 0)$.

1.1.4 Lemma *The set of restrictions to $[a,b]$ of the functions $(X-\tau_i)_+^j$ $(0 \le i \le \ell-1,\ r_i \le j \le k)$ is a basis of $\mathcal{P}_{k,\tau,r}$.*

Proof Since $(X-\tau_i)_+$ is of class $\mathcal{C}^o$, it is clear by induction on j that $(X-\tau_i)_+^j$ is of class $\mathcal{C}^{j-1}$ at τ_i, so that $(X-\tau_i)_+^j \in \mathcal{P}_{k,\tau,r}$ for $r_i \le j \le k$; as there are $(k+1)\ell - \sum_{i=1}^{\ell-1} r_i$ such functions, it is enough to see that the $(X-\tau_i)_+^j$'s are linearly independent, which is clear.

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1.2 AN EXAMPLE: CUBIC SPLINES

Let $\mathcal{S}_{k,\tau}$ denote the space $\mathcal{P}_{k,\tau,r}$ where $r_i = k$ $(1 \le i \le \ell-1)$; traditionally $\mathcal{S}_{k,\tau}$ is called the space of splines of degree k with knots τ_i; one then has

$$\dim \mathcal{S}_{k,\tau} = \ell(k+1) - \sum_{i=1}^{\ell-1} k = k + \ell,$$

and the splines are $\mathcal{C}^{k-1}$ functions.

The most usual case is that of cubic splines.

1.2.1 Proposition *Let* $M_i = (\tau_i, y_i)$ $(0 \le i \le \ell)$ *be* $\ell + 1$ *points in* $\mathbf{R}^2$*; if two numbers* α *and* β *are given, there exists a unique spline function* $f \in \mathcal{S}_{3,\tau}$ *such that the curve* $y = f(x)$ *passes through the points* M_i *and satisfies*

$$\begin{cases} f'(a) = \alpha \\ f'(b) = \beta. \end{cases}$$

Proof This proposition is introductory so we will give the proof only in the (easier) case of a uniform spacing of the knots τ_i $(\tau_{i+1} - \tau_i = h)$. The above conditions give $\ell + 3$ linear conditions

$$\begin{cases} f(\tau_i) &= y_i \qquad (0 \le i \le \ell) \\ f'(a) &= \alpha \\ f'(b) &= \beta. \end{cases}$$

The dimension of $\mathcal{S}_{3,\tau}$ is $\ell + 3$, so it is enough to see that these conditions are linearly independent.

For that, it is convenient to write the restriction P_i of f to $[\tau_i, \tau_{i+1}]$ in the following way

$$P_i(u) = a_i + b_i u + c_i u^2 + d_i u^3 \quad (0 \le i \le \ell - 1)$$

with $0 \le u \le 1$, $u = \frac{t - \tau_i}{h}$, and

$$\begin{cases} P_i(0) = f(\tau_i) = y_i \\ P_i(1) = f(\tau_{i+1}) = y_{i+1} \qquad (0 \le i \le \ell - 1). \end{cases}$$

The conditions on a_i, b_i, c_i, d_i are then

(1) $$a_i = y_i$$

(2) $$a_i + b_i + c_i + d_i = y_{i+1} \qquad (0 \le i \le \ell - 1)$$

(with $f(\tau_i) = y_i$)

(3) $$\begin{cases} b_o = \alpha/h \\ b_{\ell-1} + 2c_{\ell-1} + 3d_{\ell-1} = \beta/h \end{cases}$$

(boundary conditions), and the fact that the function is C^2

(4) $$b_i + 2c_i + 3d_i = b_{i+1}$$

(5) $$2c_i + 6d_i = 2c_{i+1} \quad (1 \leq i \leq \ell - 2)$$

which gives immediately by (1), (2) and (4)

$$\begin{cases} c_i &= 3(y_{i+1} - y_i) - 2b_i - b_{i+1} \\ d_i &= -2(y_{i+1} - y_i) + b_i + b_{i+1} \end{cases}$$

and upon inserting these formulas in (5) (for index $i+1$),

$$b_{i-1} + 4b_i + b_{i+1} = 3(y_{i+1} - y_{i-1}) \qquad (1 \leq i \leq \ell - 1)$$

which gives the following system to compute the b_i's

$$\begin{pmatrix} 1 & & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 1 & 4 \end{pmatrix} \begin{pmatrix} b_o \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ b_{\ell-1} \end{pmatrix} = \begin{pmatrix} \alpha/h \\ 3(y_2 - y_o) \\ 3(y_3 - y_1) \\ \cdot \\ \cdot \\ 3(y_\ell - y_{\ell-2}) - \beta/h \end{pmatrix}.$$

The matrix of this system is clearly nonsingular, as is easily seen by induction (one may also apply Theorem 3.1.1). The fact that the principal minors are all non zero implies that one may apply the Gauss method without pivoting, i.e., that at the i^{th} step, the i^{th} element of the i^{th} row is non zero and so may be taken as the i^{th} pivot.

One sees also that the solution of this system is very fast: the number of operations required is linear in ℓ.

The use of the word spline comes from the following theorem.

1.2.2 Theorem *Let $f \in S_{3,\tau}$ be the spline function such that*

$$\begin{cases} f(\tau_i) = y_i & (0 \leq i \leq \ell) \\ f'(a) = \alpha & \\ f'(b) = \beta & (\text{see } 1.2.1). \end{cases}$$

Then if E denotes the set of C^r functions on $[a,b]$ such that

$$\begin{cases} \phi(\tau_i) = y_i & (0 \le i \le \ell) \\ \phi'(a) = \alpha \\ \phi'(b) = \beta \end{cases}$$

f is the only element of E which minimizes (among the elements of E) the integral $\int_a^b [\phi'']^2(t)\,dt$.

Proof Let $\phi \in E$; set $e = \phi - f$ ("error in the approximation of ϕ by the spline function f").

Let $S_{1,\tau}$ be the space of the "splines of degree one on $[a,b]$ with knots at τ_i", i.e., the C^o functions, piecewise linear, with change of slope at the points τ_i.

1.2.3 Lemma

$$\int_a^b e''(x)\,h(x)\,dx = 0 \quad \forall h \in S_{1,\tau}$$

Proof

$$\int_a^b e''(x)\,h(x)\,dx = \sum_{i=0}^{\ell-1} \int_{\tau_i}^{\tau_{i+1}} e''(x)\,h(x)\,dx$$

$$= \sum_{i=0}^{\ell-1} \left((e'h)_{\tau_i}^{\tau_{i+1}} - \int_{\tau_i}^{\tau_{i+1}} e'h' \right)$$

(integration by parts). But we have

$$\sum_{i=0}^{\ell-1} (e'h)_{\tau_i}^{\tau_{i+1}} = e'h(b) - e'h(a) = 0$$

because $e'(a) = e'(b) = 0$ by the definition of E, and

$$\sum_{i=0}^{\ell-1} \int_{\tau_i}^{\tau_{i+1}} e'h' = \sum_{i=0}^{\ell-1} \lambda_i (e(\tau_{i+1}) - e(\tau_i)) \quad \text{if } h'(x) = \lambda_i \quad \text{for } x \in]\tau_i, \tau_{i+1}[,$$

and we have $e(\tau_i) = 0$, as by definition, $f(\tau_i) = \phi(\tau_i) \quad (0 \le i \le \ell)$.

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Completion of the proof of Theorem 1.2.2

Let $\phi \in E$, $e = \phi - f$; we have

$$\int_a^b \phi''^2 = \int_a^b (e'' + f'')^2 = \int_a^b e''^2 + \int_a^b f''^2 + 2\int_a^b e''f''.$$

But since f is in $\mathcal{S}_{3,\tau}$, one has $f'' \in \mathcal{S}_{1,\tau}$, and then $\int_a^b e''f'' = 0$ by Lemma 1.2.3; one thus has $\int_a^b \phi''^2 \geq \int_a^b f''^2$, and if there is equality, one deduces $\phi'' = f''$ (as then $\int_a^b e''^2 = 0$ and so $e'' = 0$ because e'' is continuous), which implies immediately $\phi = f$ if $\ell \geq 1$, because ϕ and f satisfy $\phi'(\alpha) = f'(\alpha)$, $\phi'(\beta) = f'(\beta)$ and $\phi(\tau_i) = f(\tau_i)$.

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1.3 B-SPLINES: FUNDAMENTAL PROPERTIES

We now generalize the spline functions of Section 1.2, by considering the space $\mathcal{P}_{k,\tau,r}$ (see 1.1). In other words,

a) k will take any value (i.e., not necessarily 3),

b) the functions will be of class $\mathcal{C}^{r_i-1}$ at the point τ_i $(1 \leq i \leq \ell - 1)$ with $r_i \leq k$ (and not necessarily $r_i = k$).

To be able to describe a manageable base of this space, we shall first define B-spline functions.

1.3.1 Notation Let us take in $\mathbf{R}$ a sequence $t_0, \ldots, t_m$ of points called *knots* such that for all i, $t_i \leq t_{i+1}$. If there are r t_i's equal to τ, one says that τ is a node *of order* r or of *multiplicity* r.

Moreover, for $1 \leq j \leq m + 1 - i$, one sets:

$$\omega_{i,j}(x) = \begin{cases} \dfrac{x - t_i}{t_{i+j} - t_i} & \text{if} \quad t_i < t_{i+j} \\ 0 \quad \text{otherwise.} \end{cases}$$

1.3.2 Definition of B-splines

With the above notation, set $t = (t_0, \ldots, t_m)$. *For* $x \in \mathbf{R}$, $0 \leq i \leq m - k - 1$, *the functions* $B_{i,k,t}(x)$ *(also denoted* $B_{i,k}(x)$ *when* *the sequence t is fixed) are defined by induction on k in the following way*

$$\begin{cases} B_{i,0}(x) = \begin{cases} 1 & \text{if} \quad t_i \leq x < t_{i+1} \\ 0 & \quad \text{otherwise} \end{cases} \\ \\ B_{i,k}(x) = \omega_{i,k}(x) B_{i,k-1}(x) \ + \ (1 - \omega_{i+1,k}(x)) B_{i+1,k-1}(x) \ \text{ for } k \geq 1. \end{cases}$$

By definition of $\omega_{i,k}(x)$, one has

$$B_{i,k}(x) \ = \ \frac{x - t_i}{t_{i+k} - t_i} B_{i,k-1}(x) \ + \ \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} B_{i+1,k-1}(x)$$

if $t_i < t_{i+k}$ and $t_{i+1} < t_{i+k+1}$.

1.3.3 Remarks

1) One may define in the same way B-splines for an *infinite* sequence of knots t_i $(t_i \leq t_{i+1})$, since the definition of each $B_{i,k}$ uses only a finite numbers of knots (see Proposition 1.3.4 below: the spline $B_{i,k}(x)$ has $[t_i, t_{i+k+1}]$ for support, and its definition uses only those t_j such that $i \leq j \leq i + k + 1$).

2) If, for an index i, $t_i = t_{i+k+1}$ (and so t_i is a node of multiplicity $\geq k + 2$), one has $B_{i,k} \equiv 0$; the converse is also true: see 1.3.4 below.

1.3.4 Proposition *With the above notation, one has the following properties:*

a) *$B_{i,k}(x)$ is a piecewise polynomial of degree k.*

b) *$B_{i,k}(x) = 0$ for $x \notin [t_i, t_{i+k+1}[$.*

c) *$B_{i,k}(x) > 0$ for $x \in]t_i, t_{i+k+1}[$; $B_{i,k}(t_i) = 0$ unless $t_i = t_{i+1} = \cdots = t_{i+k} < t_{i+k+1}$ (node of order $k+1$), and then $B_{i,k}(t_i) = 1$.*

d) *Let $[a, b]$ be an interval such that $t_k \leq a$, $t_{m-k} \geq b$. Then $\sum_{i=0}^{m-k-1} B_{i,k}(x) = 1$ for all $x \in [a, b[$.*

e) *Let $x \in]t_i, t_{i+k+1}[$. Then $B_{i,k,t}(x) = 1$ if and only if $t_{i+1} = \cdots = t_{i+k} = x$.*

f) *$B_{i,k}(x)$ is right-continuous (and even right-infinitely differentiable), for all $x \in \mathbf{R}$ (recall that $B_{i,k}(x) = 0$ for x outside $[t_0, t_m]$).*

1.3.5 Remark One sees already in this proposition some remarkable properties of the B-splines: d) shows that they constitute a *partition of* *unity* on a convenient interval, and b) shows that each $B_{i,k}(x)$ has a "*small support*".

Proof Properties a), b), c), d) and f) are clear for $k=0$; one deduces immediately a), b), c) and f) for $B_{i,k}$ by induction on k.

Let us prove property d) by induction on k. Let $x \in [a,b]$. There exists then j, $j \geq k$, $j \leq m-k-1$, such that $x \in [t_j, t_{j+1}[$.

If $x = t_j$ and $B_{j,k}(x) = 1$, the property is clear (see c)).

In the other cases, one has

$$\sum_{i=0}^{m-k-1} B_{i,k}(x) = \sum_{i=j-k}^{j} B_{i,k}(x)$$

by b), and

$$\sum_{j-k}^{j} B_{i,k}(x) = \sum_{j-k}^{j} \omega_{i,k} B_{i,k-1}(x) + \sum_{j-k}^{j} (1-\omega_{i+1,k}) B_{i+1,k-1}(x)$$

by the definition of the B-splines, which implies, grouping terms together,

$$\sum_{i=j-k}^{j} B_{i,k}(x) = \omega_{j-k} B_{j-k,k-1}(x) + \sum_{i=j+1-k}^{j} B_{i,k-1}(x) + (1-\omega_{j+1,k}) B_{j+1,k-1}(x).$$

But $B_{j-k,k-1}(x) = 0$ and $B_{j+1,k-1}(x) = 0$ because $x \in [t_j, t_{j+1}[$ (see b)) and

$$\sum_{i=j+1-k}^{j} B_{i,k-1}(x) = 1$$

by the induction hypothesis, which implies d).

e) is an easy consequence of d) and c).

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1.3.6 Remark If $t_{m-k} = \cdots = t_m = b$, formula d) is valid only on the interval $[a,b[$, because for all i $(1 \leq i \leq m-k-1)$, one has $B_{i,k}(b) = 0$ (see f) above: B-splines are right-continuous). If we want a formula valid on $[a,b]$, we have to set $B_{m-k-1,k}(b) = 1$, which makes the B-spline $B_{m-k-1,k}(x)$ *left continuous* at b.

We will systematically use this convention, which is necessary only if there is a knot of multiplicity $k+1$ at b.

1.3.7 Examples

a) $k = 1$

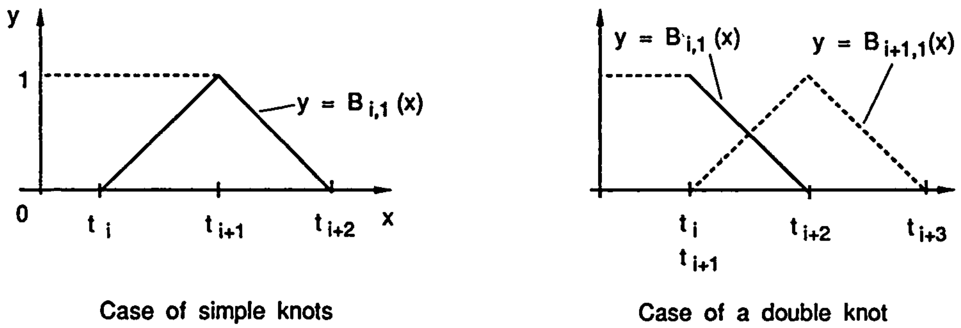


Figure 1.3.1

b) $k = 2$ (i.e., piecewise quadratic functions)

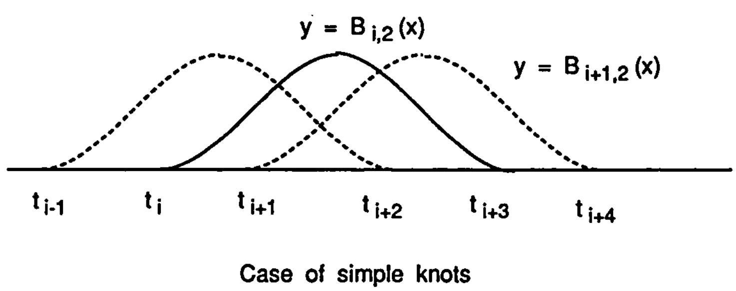
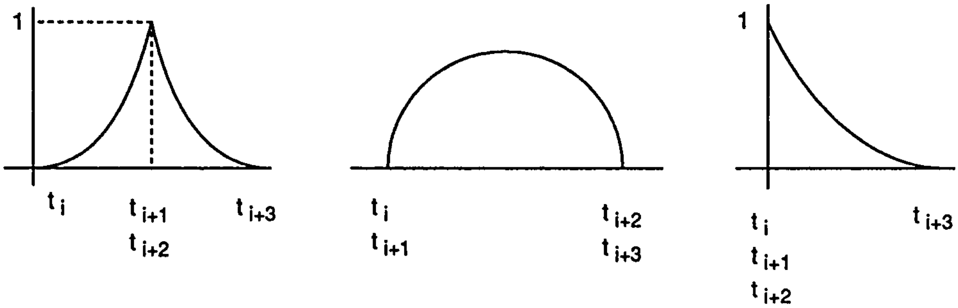


Figure 1.3.2

(The curve $y = B_{i,2}(x)$ is built from three arcs of parabolas and two half-lines with junction $\mathcal{C}^1$ at the knots t_i, t_{i+1}, t_{i+2} and t_{i+3}).



$y = B_{i,2}(x)$ in the case of multiple knots

Figure 1.3.3

c) Let us represent the 9 elements of the B-spline basis of $\mathcal{P}_{3,t}$ when $n = 9$, i.e., with 5 knots (simple) in $]a,b[$, and $t_0 = \cdots = t_3 = a$ and $t_9 = \cdots = t_{12} = b$

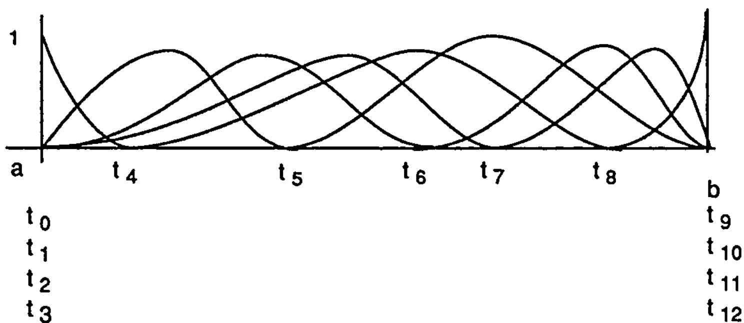


Figure 1.3.4

1.3.8 Remark We will give later (Chap. 3, Section 3.3) the expressions for the B-splines $B_{i,3}(x)$ in the case of an uniform sequence of knots .