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1
The Petersen Graph

0. Prologue

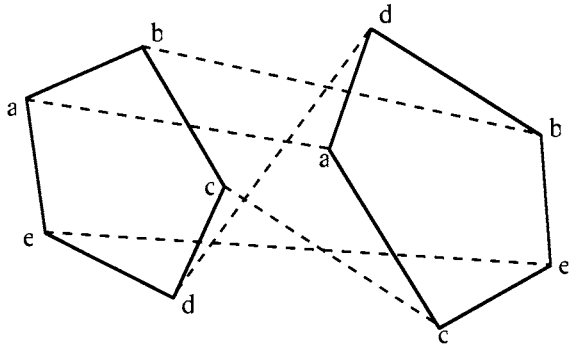
The index of Capobianco and Molluzzo's book "Examples and Counterexamples in Graph Theory" [C-M 78] shows the following entry:-

"Petersen graph, 36, 54, 109, 146, 158, 164, 179, 213"

This makes it by far the most referenced graph in that volume. It also gives some idea of the importance of the Petersen graph in the study of graph theory.

1. What is a graph?

The Petersen graph P got its name because of its appearance in 1898 in a paper by J. Petersen [jP 98] (see Fig.1.1). However, this was not the first time it had appeared in print. The graph first appeared in [aK 86] as the graph of the Desargues' configuration.



The Petersen graph P

Figure 1.1

It was appropriate that in [jP 98] the graph was used as a counterexample because, as we shall see, it continues to be a yardstick with which to measure conjectures. It also has many interesting properties. These two facts give it a unique place in the theory of graphs.

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In this chapter we will introduce basic graph theoretic properties and show how they apply to the Petersen graph, P. The remainder of the book will develop a number of areas in which P plays an important role.

But what is a graph?

Precisely, it is an ordered pair G = (VG,EG) where VG is the non-empty set of **vertices** of G, and EG, the set of **edges** of G, is a subset of the 2-subsets of VG. That is, the edges of G are sets of size two, whose members are elements of VG. Unless otherwise specified we will assume |VG| is finite.

For instance, the following are graphs.

$G_1 = (\{1,2,3\},\{\{1,2\},\{2,3\},\{3,1\}\})$

$G_2 = (\{1,2,3,4\},\{\{1,2\},\{1,3\},\{1,4\}\})$.

When dealing with examples it is frequently valuable to give graphs a pictorial representation. This is done by letting the vertices of a graph G be represented by dots and the edges by lines joining the relevant dots.

The graphs G_1 and G_2 are shown in visual representations in Figure 1.2.

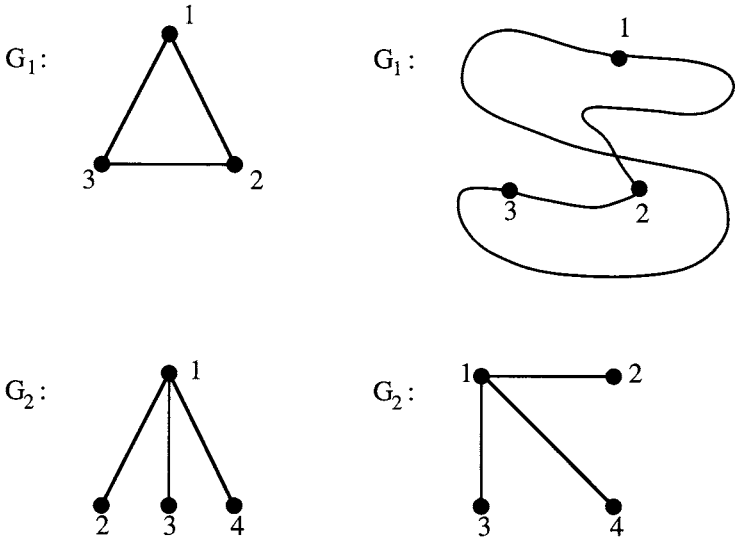

Figure 1.2

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The moral to be drawn from Figure 1.2 is that there is no **unique** way of drawing a graph. Vertices may be placed arbitrarily in the plane and edges may be drawn in any convenient way. Generally we will use straight lines for edges but in the end we will be guided in the main by aesthetics and the purpose in hand.

In general we will use uv for the edge $\{u,v\}$ as this will avoid a surfeit of braces. Further, in such a case we say that u and v are **adjacent**, the vertices u,v are **incident** with uv, and uv is **incident** with u and v. We sometimes write $u \sim v$ to show that u is adjacent to v and $u \not\sim v$ when $uv \notin EG$.

With this in mind the Petersen graph P, has $VP = \{1,2,3,4,5,1',2',3',4',5'\}$ and $EP = \{12,23,34,45,51,1'3',2'4',3'5',1'4',5'2',11',22',33',44',55'\}$. We show one of its common pictorial representations in Figure 1.3. Unless stated otherwise, we will always refer back to this drawing and labelling of P. It will sometimes be useful to let $A = \{12,23,34,45,51\}$, $B = \{11',22',33',44',55'\}$ and $C = \{1'3',2'4',3'5',4'1',5'2'\}$.

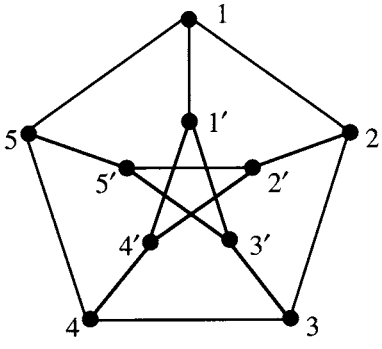

Figure 1.3

Because uv is a 2-subset of VG, then u and v are distinct vertices. Occasionally we will want to discuss 'edges' which join u to itself. These we refer to as **loops.**

Again, we defined EG to be a subset of the 2-subsets of VG. Hence there is at most one edge incident with a given pair of vertices. Sometimes we allow multiple edges between a given pair of vertices but in that case we will call the 'graphs' involved, **multigraphs**.

If we allow multigraphs to have loops, then they will be called **pseudographs**. We note that what we have defined here as graphs are elsewhere often called **simple graphs**.

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A graph, multigraph and pseudograph on 5 vertices are shown in Figure 1.4.

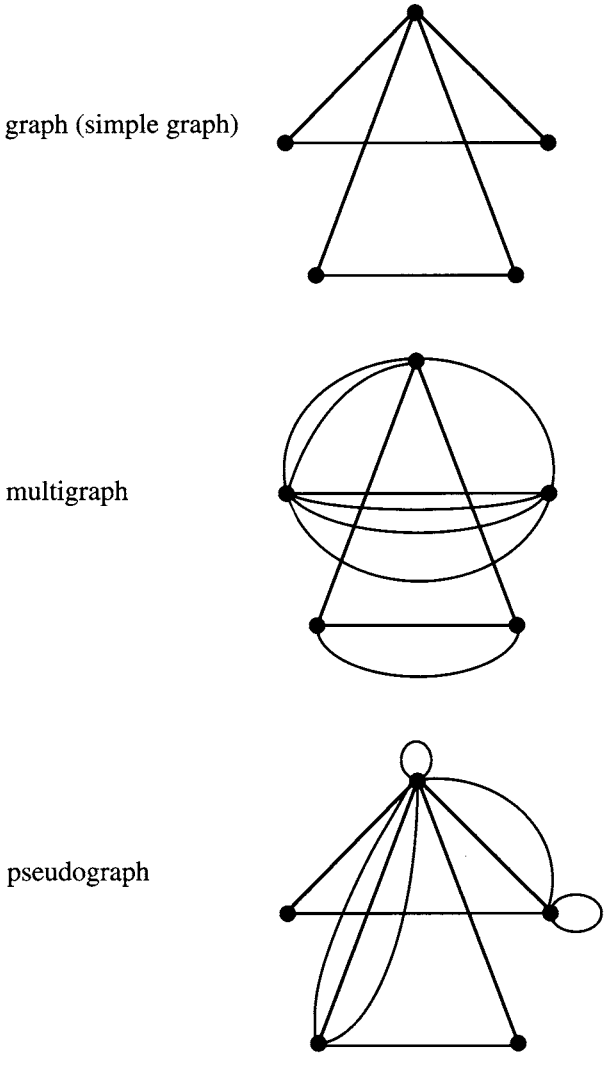

Figure 1.4

We also define a **eunegraph**. This is a pseudograph which may also contain "incomplete" edges or **splines**. That is, edges that are incident with only one vertex. We will usually convert eunegraphs to pseudographs by joining the splines to make edges, loops or multiple edges. A eunegraph will be represented as in Figure 1.5, where vertices u and v are incident to splines.

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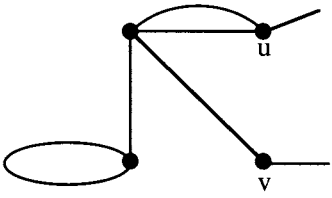

Figure 1.5

The term eunegraph is coined here for the first time. However it is a concept that we will find valuable in later chapters. Graphs with incomplete edges have proved useful in a number of places in the literature.

It is important to be able to decide when two graphs are the same.

Recall the graph G_1 of Figure 1.2. Clearly if we relabel G_1, then we should obtain essentially the same graph. Hence $G_3 = (\{u,v,w\},\{uv, uw, wv\})$ is the 'same' as G_1. We therefore say that graphs G and H are **isomorphic** if there exists a bijection $\varphi: VG \to VH$ which is also a bijection from EG to EH when we define $\varphi(uv) = \varphi(u)\varphi(v)$. The mapping φ is an **isomorphism** between G and H. We write $G \cong H$ for 'G is isomorphic to H'.

In Figure 1.6 we give four isomorphic graphs on six vertices.

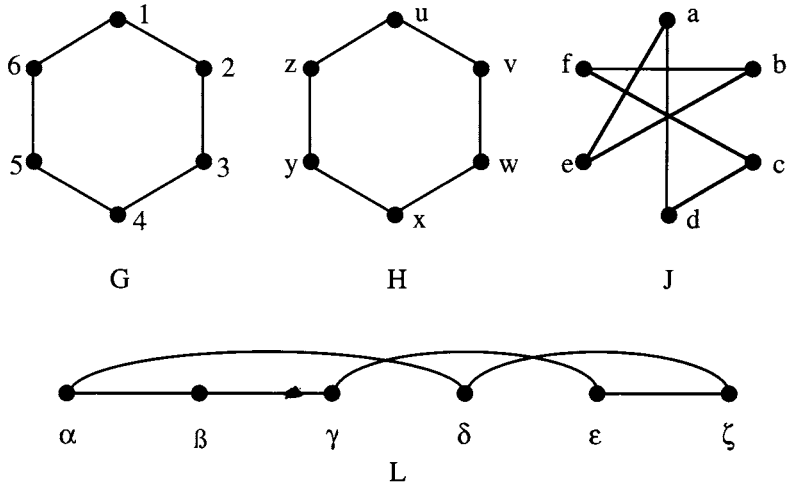

Figure 1.6

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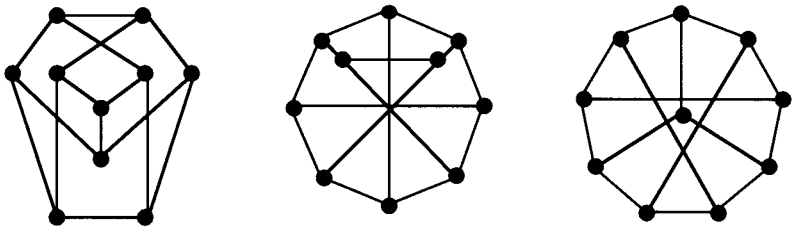
For instance, an isomorphism φ between G and L can be expressed as follows:-

$\varphi(1) = \alpha$ $\varphi(2) = \beta$ $\varphi(3) = \gamma$ $\varphi(4) = \varepsilon$ $\varphi(5) = \zeta$ $\varphi(6) = \delta$.

It is straightforward to check that φ is a bijection on the edges of G and L.

Figure 1.7 shows three isomorphic copies of the same graph. These are in fact three more ways to draw P.



(a) (b) (c)

Figure 1.7

Now isomorphism is an equivalence relation on the set of all graphs. Consequently when we talk about a graph we are usually referring to it as a representative of its equivalence class. As such any labelling or names we assign to the vertices are for ease of manipulation. Unlike a road map of a country where the vertices represent towns and cities and the edges represent roads, no special significance will be attached to this labelling. If, as in the road map, there **is** good reason for assigning certain names to vertices, then we will call the graph a **labelled graph**.

In Figure 1.8 we give all the graphs on three vertices, as well as all the labelled graphs on three vertices.

Define the **order** of G to be $|VG|$ and the **size** of G to be $|EG|$. Then the graphs of Figure 1.8 have order 3 and the size of graph J in Figure 1.6 is 6 and equals its order. Further, the order of the Petersen graph is 10 and its size is 15.

Now it is relatively easy to determine the number of labelled graphs of order n, but the number of graphs of order n needs special techniques involving Pólya's Theorem, which we will not go into in this book; see [H-P 73].

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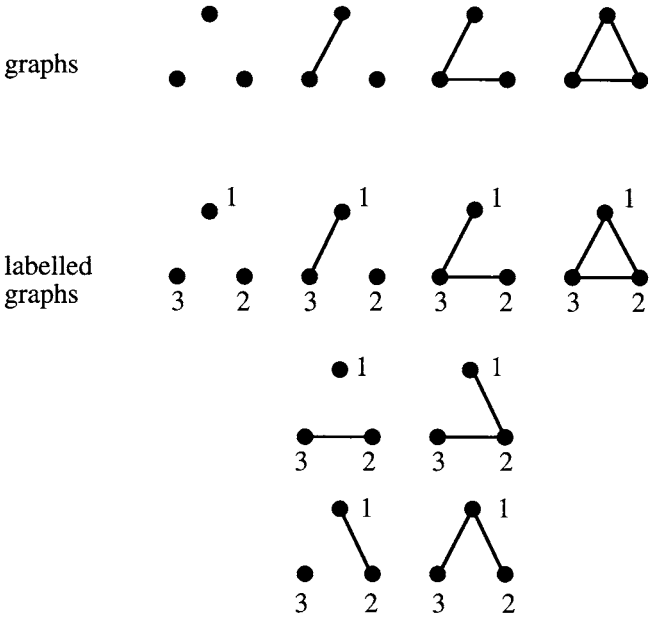

Figure 1.8

The number of edges in G incident with a vertex u (plus twice the number of loops at u, in the case of a pseudograph) is called the **degree** of u and is written $\deg_G u$. Thus $\deg_{G_2} 1 = 3$ and $\deg_{G_2} 2 = 1$ for the graph G_2 of Figure 1.2. Every vertex of P has degree 3.

Where there is no ambiguity we drop the subscript G and simply write $\deg u$. The smallest, respectively largest, degree of any vertex in a graph G is denoted by $\delta(G)$, respectively $\Delta(G)$.

We are now able to state and prove our first result in graph theory.

Lemma 1.1 $$\sum_{u \in VG} \deg u = 2|EG|.$$

Proof: The edge uv is counted twice in the summation, once in deg u and once in deg v. Hence the result follows. □

Corollary 1.2 Every graph has an even number of vertices of odd degree. □

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A **neighbour** of a vertex $u \in VG$ is any member of the set $\{v \in VG : uv \in EG\}$, which is called the **neighbourhood of u**, $N_G(u)$. (If there is no ambiguity we write simply $N(u)$.) We note that $|N(u)| = \deg u$.

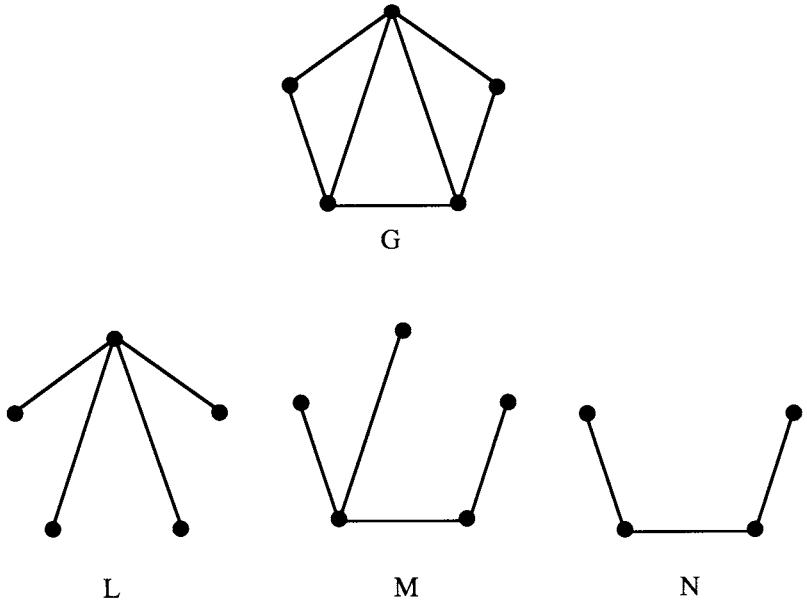

Figure 1.9

Another common graph-theoretical notion is that of subgraph. We say that H is a **subgraph** of G if $VH \subseteq VG$ and $EH \subseteq EG$. A graph and some of its subgraphs are shown in Figure 1.9.

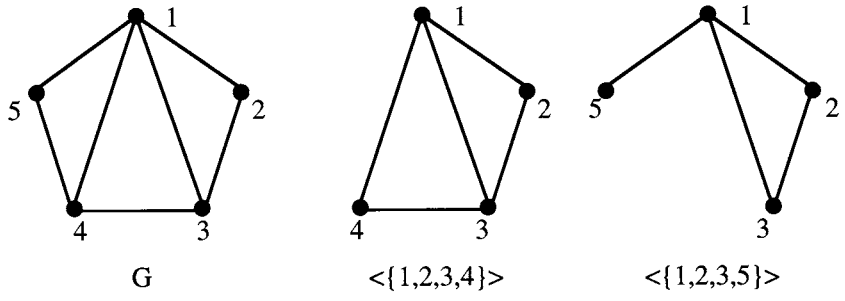

Figure 1.10

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Two special types of subgraph are worth noting. We say that H is an **induced** subgraph of G, if every pair of vertices of H that are adjacent in G, are adjacent in H. If $VH = X \subseteq VG$, we write $< X >$ for the subgraph of G induced by the vertex set X. Figure 1.10 gives two examples of induced subgraphs. The labelling of G is given for ease of description.

A special case of an induced subgraph is the **vertex deleted subgraph**. This is defined to be $G_v = <VG \setminus \{v\}>$. It is the graph obtained from G by removing the vertex v and all edges of G that were incident with v. See Figure 1.11 for two examples, one of which is isomorphic to the subgraph N of Figure 1.9 and the other to the subgraph $<\{1, 2, 3, 5\}>$ of Figure 1.10.

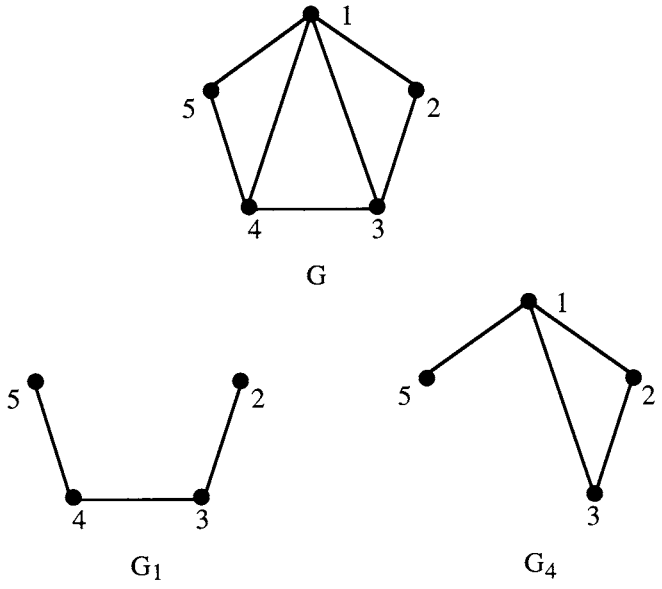

Figure 1.11

Rather than G_v we sometimes write G - v. We define G - {u,v} to be the graph $(G_u)_v$ (or alternatively the graph (G - u) - v).

Figure 1.12 (a), (b) shows the vertex deleted subgraphs of P obtained by deleting vertex 1 and vertex 1′. By the symmetry of the drawing of P in Figure 1.3, $P_{1'} \cong P_{2'} \cong P_{3'} \cong P_{4'} \cong P_{5'}$. It is not difficult to show though that $P_1 \cong P_{1'}$. Hence there is essentially only one vertex deleted subgraph of P.

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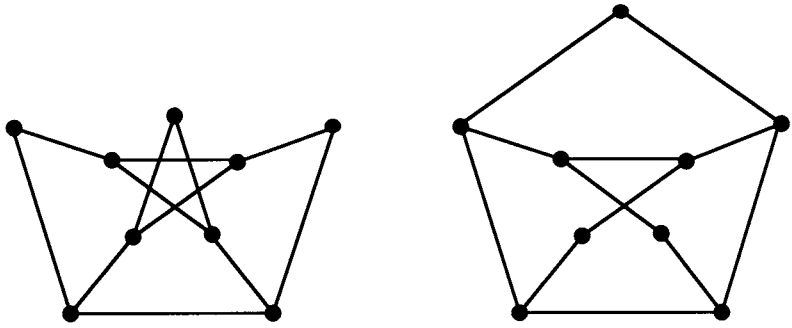


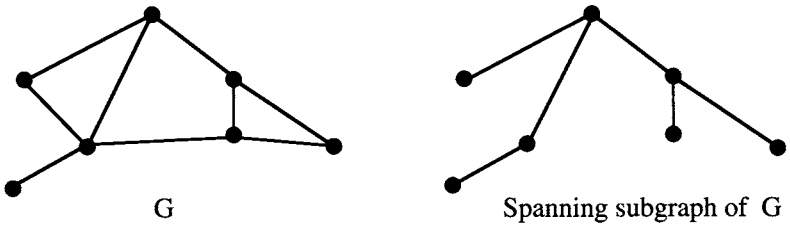
(a) P_1 (b) $P_{1'}$

Figure 1.12

There is an important open question in graph theory which deals with vertex deleted subgraphs. This is Ulam's Conjecture or the Reconstruction Conjecture, which says that any graph can be determined uniquely from its collection of vertex deleted subgraphs. Equivalently, $\{G_v: v \in VG\}$ is unique to the graph G.

Much work is currently being done on this conjecture. Unfortunately it is all beyond the scope of this book, but the interested reader should look at the excellent review articles by Bondy and Hemminger [B-H 77] and Nash-Williams [cN 78]. However we will have cause to consider vertex deleted subgraphs at several stages throughout this book.

The other useful type of subgraph which deserves mention at this stage is the spanning subgraph. Now H is a **spanning subgraph** of G if H is a subgraph of G with $|VH| = |VG|$. A graph and one of its spanning subgraphs are shown in Figure 1.13.



G Spanning subgraph of G

Figure 1.13

Figure 1.14 shows a spanning subgraph of P which will be of interest later. This subgraph is taken from Figure 1.7(c). The reasons for the labelling will become clear in Section 5.

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