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Group Actions and Riemann Surfaces

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1. Introduction

We begin by considering two familiar ways of constructing Riemann surfaces. First, we take a power series P converging on some disc D_0 with centre z_0, and expand P about some point z_1 in D_0 other than z_0. In general, P will converge in a disc D_1 extending beyond D_0, and if we continue this process indefinitely we obtain a maximal Riemann surface on which the analytic continuation of P is defined. Of course, if we return to a region where P is already defined, but with different values, we create a new 'sheet' of the surface; thus we are led to the notion of a Riemann surface constructed from a given power series: this is the Weierstrass approach. A more modern approach is simply to define a Riemann surface as a complex analytic manifold but either way, there is the problem of showing that these two definitions are equivalent. It is easy enough to see that a Riemann surface obtained by analytic continuation is an analytic manifold, so we must focus our efforts on showing that every analytic manifold supports an analytic function. One solution to this problem lies in showing first that every such manifold arises as the quotient by a group action, and second, that we can construct functions invariant under this group action. As a by-product of a study of these groups we obtain important and very detailed quantitative information about the geometric nature of the general Riemann surface.

In pursuing this line of thought we are led naturally into the study of discrete group actions on the three classical geometries of constant curvature (the sphere, the Euclidean plane and hyperbolic plane), and also to the way that discreteness imposes severe geometric constraints on the corresponding quotient surfaces. The crystallographic restriction on Euclidean groups is one example of this, but, as we shall see, this idea reaches out well beyond the study of wall-paper patterns and goes on to exert a powerful influence on

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the geometric structure of all Riemann surfaces. It is this influence that we attempt to describe in this essay.

2. The Uniformisation Theorem

The Uniformisation Theorem implies that every Riemann surface can be realised as the quotient by a group action and, once we have given an explicit description of the possible groups that arise in this way, this lays bare the geometric nature of Riemann surfaces for all to see. In this section we discuss the group actions that arise and later we shall show how, by examining these, we can obtain universal information about the metric and the geometric structure of the general Riemann surface.

Consider now an arbitrary Riemann surface $\mathcal{R}$. It is a straightforward matter to construct the topological universal covering surface $\mathcal{S}$ and to lift the conformal structure from $\mathcal{R}$ to $\mathcal{S}$, thus realising $\mathcal{R}$ as the quotient of the simply connected Riemann surface $\mathcal{S}$ by the corresponding cover group G. We know (from topology) that G has certain interesting properties (for example, only the identity in G can have fixed points on $\mathcal{S}$, and G is the fundamental group of $\mathcal{R}$) but, so far, we have little concrete information about $\mathcal{S}$ or G.

The key step now is to invoke the strong form of the *Riemann Mapping Theorem*, namely that every simply connected Riemann surface is conformally equivalent to one of the three spaces

(a) the extended complex plane (or the Riemann sphere) $\mathbb{S}$,
(b) the complex plane $\mathbb{C}$,
(c) the hyperbolic plane $\mathbb{H}$ (that is, the unit disc $\{z : |z| < 1\}$ in $\mathbb{C}$).

With this available, we can pass (by a conformal mapping) from the universal cover $\mathcal{S}$ to one of these spaces and henceforth take one of $\mathbb{S}$, $\mathbb{C}$ or $\mathbb{H}$ as the universal cover of $\mathcal{R}$. We can immediately draw some interesting conclusions from this representation; for example, it shows that every Riemann surface has a countable base, and also an exhaustion by compact sets. (We remind the reader that there are surfaces which do not have these desirable properties.)

The benefits of using only $\mathbb{S}$, $\mathbb{C}$ or $\mathbb{H}$ as universal covering spaces are enormous and these arise from the (easily proved) analytic fact that the group of conformal automorphisms of each of these spaces is a subgroup of the Möbius group $\mathcal{M}$ consisting of all maps of the form

$$g(z) = \frac{az+b}{cz+d}, \quad ad - bc = 1. \tag{1}$$

In fact, $\mathcal{M}$ is the full group of conformal automorphisms of $\mathbb{S}$, the conformal automorphisms of $\mathbb{C}$ are the maps $g(z) = az + b$, and conformal automorphisms of $\mathbb{H}$ are

$$g(z) = \frac{az+\bar{c}}{cz+\bar{a}}, \quad |a|^2 - |c|^2 = 1. \tag{2}$$

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We have now reduced the study of the general Riemann surface (defined as a manifold) to the study of quotients by discrete subgroups of $\mathcal{M}$. Of course, each of the three spaces $\mathbb{S}$, $\mathbb{C}$ or $\mathbb{H}$ are metric spaces, the hyperbolic metric on $\mathbb{H}$ being derived from the line element

$$ds = \frac{2|dz|}{1-|z|^2},$$

and we shall be particularly interested in the Möbius isometries of these spaces. In the case of $\mathbb{H}$, all automorphisms of the form (2) are hyperbolic isometries and these are the only conformal isometries of $(\mathbb{H}, \rho)$. For more information, see [2], [4], [5], [7], [11] and [12].

Each Riemann surface $\mathcal{R}$ determines its universal cover uniquely from among the three possibilities $\mathbb{S}$, $\mathbb{C}$ or $\mathbb{H}$ because these three spaces are conformally inequivalent. There are very few surfaces $\mathcal{R}$ which have $\mathbb{S}$ or $\mathbb{C}$ as their universal cover and we can easily dispose of these. If $\mathcal{R}$ has $\mathbb{S}$ as its universal cover, the cover group G is trivial (for only the identity in G can have fixed points on $\mathbb{S}$) and so (up to conformal equivalence) $\mathcal{R} = \mathbb{S}$. This shows that there is only one conformal structure on the sphere.

The case when $\mathcal{R}$ has $\mathbb{C}$ as its universal cover is hardly more interesting. As the automorphism group of $\mathbb{C}$ is the class of maps $z \mapsto az+b$, the requirement that only the identity in G can have fixed points on $\mathbb{C}$ means that G must be a group of translations (that is, $a = 1$). As G is also discrete, we see that either G is the trivial group (and $\mathcal{R} = \mathbb{C}$), or G is generated by one or two (linearly independent) translations. If G is a cyclic group then (up to conjugacy) G is generated by $z \mapsto z + 2\pi i$, the quotient map is $z \mapsto \exp z$ (for $\exp z = \exp w$ if and only if $w = g(z)$ for some g in G), and in this case $\mathcal{R}$ is the punctured plane $\mathbb{C} - \{0\}$. It is of interest to note that the surfaces $\mathbb{S}$, $\mathbb{C}$ and $\mathbb{C} - \{0\}$ which we have obtained so far are, collectively, the sphere with at most two punctures. In the remaining case, $\mathcal{R}$ has $\mathbb{C}$ as its universal cover, G is generated by two independent translations and $\mathcal{R}$ is a torus. Before moving on, we note that although any two tori are topologically equivalent, there are infinitely many distinct conformal equivalence classes of tori so, as far as the analyst is concerned, the discussion of this can go much further.

Our discussion so far has led to the conclusion that essentially every Riemann surface has the hyperbolic plane $\mathbb{H}$ as its universal cover and, as the local geometry of $\mathbb{H}$ projects down to $\mathcal{R}$, we see that *the intrinsic geometry of the generic Riemann surface is hyperbolic*; for example, if T is a small triangle on a generic Riemann surface then its angle sum is strictly less than π. This, of course, applies equally well to planar Riemann surfaces, so that when we study plane domains with the induced Euclidean structure in elementary complex analysis we are, in fact, using the wrong geometry. As a simple indication that we should have anticipated this, we observe that in any reasonable metric

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tied to a domain D, the boundary of D should be infinitely far from any point inside it: this is true in the intrinsic hyperbolic geometry of D but not, of course, generally true in the Euclidean metric. To be more precise, in any plane domain the hyperbolic metric is a Riemannian metric given by $ds = \lambda(z)|dz|$ where $\lambda(z) \to \infty$ as $z \to \partial D$. For 'most' domains, and for all simply connected domains, $\lambda(z)$ is of the same order as the reciprocal of the Euclidean distance of z from ∂D. There are, however, domains in which $\lambda(z) \to \infty$ at a slower rate than this as z approaches particular boundary points of D.

The realisation that the intrinsic geometry in complex analysis is hyperbolic has profound repercussions and we illustrate this here with one fundamental, but simple, result. One of the results in a standard complex analysis course is Schwarz's Lemma: if f is an analytic map of the unit disc into itself, and if $f(0) = 0$, then $|f(z)| \leq |z|$ with equality if and only if f is a rotation. Now the hypotheses asserts that f is a self-map of the hyperbolic plane into itself, and the conclusion is stated (rather perversely) in terms of the Euclidean distance $|z|$ between z and 0. However, if ρ is the hyperbolic distance in $\mathbb{H}$ the conclusion can also be stated as

$$\rho(f(z), f(0)) \leq \rho(z, 0)$$

and, with a little (but not much) extra work, we obtain the far more penetrating form known as the Schwarz-Pick Lemma: *if f is an analytic map of the hyperbolic plane into itself, then f is either a contraction or an isometry*, [1]. Many results in complex analysis depend as much on this fact as they do on analyticity.

It is now evident that we must examine discrete groups of hyperbolic isometries, and we recall that any conformal isometry of $\mathbb{H}$ is of the form

$$g(z) = \frac{az + \bar{c}}{cz + \bar{a}}, \quad |a|^2 - |c|^2 = 1.$$

A discrete group is necessarily countable, and the condition for discreteness is equivalent to the statement that $|a| \to \infty$ or, equivalently, to $|c| \to \infty$ as g runs through G. As $|a|^2 - |c|^2 = 1$, this is equivalent to $|a/c| \to 1$ and, as $|c/a| = |g(0)|$, we see that discreteness is equivalent to saying that all G-orbits accumulate only on the ideal boundary of $\mathbb{H}$ (that is, on the unit circle $|z| = 1$). We shall return to discuss the isometries in greater detail later.

We end this section with the remark that the Uniformisation Theorem is a deep result (as must be evident from the results that flow from it), and the most difficult part of the proof is the existence part, namely the Riemann Mapping Theorem. The usual proof of this involves potential theory (where the existence of the Green's function is the main issue) and we refer the reader to, for example, [1] and [3] for the details. It is, perhaps, worth mentioning

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that the existence or otherwise of a Green's function is a problem in real analysis.

3. Automorphic functions

If G is a group acting on a space X, then the class of maps $F : X/G \to Y$ corresponds precisely to the class of G-periodic maps $f : X \to Y$, that is, to maps f with the property that $f(gx) = f(x)$ for every x in X and every g in G. As an example, each analytic map $f : \mathbb{C} \to \mathbb{C}$ with period 2π corresponds to an analytic map $F : \mathbb{C} - \{0\} \to \mathbb{C}$ which has a Laurent expansion

$$F(\zeta) = \sum_{n=-\infty}^{+\infty} a_n \zeta^n$$

valid throughout $\mathbb{C} - \{0\}$. Here, the cover group G is generated by $z \mapsto z + 2\pi$, the quotient map is $z \mapsto \mathrm{e}^{iz}$ and so the periodic map f has a *Fourier expansion*

$$f(x+iy) = F(\mathrm{e}^{iz}) = \sum_{n=-\infty}^{+\infty} a_n \mathrm{e}^{inx-ny}$$

valid throughout $\mathbb{C}$ (this reduces to the usual Fourier series when $y = 0$). For more details, see [8], [9] and [10]. As an illustration of this, $z \mapsto \cos z$ corresponds to $F(\zeta) = (\zeta + \zeta^{-1})/2$. As F is a rational map of degree 2, this explains why cos is a 2-1 mapping of each fundamental strip (of width 2π) onto the sphere $\mathbb{S}$.

One of the consequences of the Uniformisation Theorem is that given any Riemann surface $\mathcal{R}$ (as a manifold), we can construct analytic functions defined on it simply by constructing analytic G-periodic functions on the appropriate covering space, and hence we can establish to equivalence of the two concepts discussed in Section 1. Of course, any function of the form $\sum_{g \in G} f(g(z))$ is periodic providing that the sum is absolutely convergent (or G is finite); however, in general the sum will diverge and some modifications are required to force convergence.

In the case of the Weierstrass elliptic function $\wp$, for example, (where $\mathcal{R}$ is a torus) the group G is generated by two independent translations $z \mapsto z + l$ and $z \mapsto z + \mu$ and one defines

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \left(\frac{1}{(ml + n\mu - z)^2} - \frac{1}{(ml + n\mu)^2} \right).$$

Without the term independent of z, the series would diverge and the subtraction of this term is designed to make the difference small enough to force absolute convergence (as indeed it does).

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Our interest centres on the generic Riemann surface with the hyperbolic plane $\mathbb{H}$ as its covering space, so our task is really to take any discrete group G of hyperbolic isometries acting on $\mathbb{H}$, and to learn how to construct periodic functions or, as they are more commonly known in the subject, *automorphic functions*. The key step is to note that, ignoring questions of convergence for the moment, if

$$\theta(z) = \sum_{g\in G} f(g(z))g'(z)^t, \tag{3}$$

then, for h in G,

$$\theta(h(z))h'(z)^t = \sum_{g\in G} f(gh(z))g'(hz)^t h'(z)^t = \sum_{\gamma\in G} f(\gamma(z))\gamma'(z)^t = \theta(z),$$

and so the quotient of any two such θ-functions (constructed using two different choices of f) will be automorphic. In fact, any function θ here gives rise to a differential on the Riemann surface: for more details, see [5], [8], [9] and [10].

We must now pay attention to the convergence. First, we impose reasonable properties on the function f in (3) (which, in this exposition, we ignore), and then attempt to obtain absolute convergence by studying the series

$$\sum_{g\in G} g'(z)^t. \tag{4}$$

Now the general isometry is given by g in (2) and

$$g'(z) = \frac{1}{(cz+\bar{a})^2} = \frac{1}{c^2(z-g^{-1}(\infty))^2}\ .$$

The point $g^{-1}(\infty)$ lies outside the unit circle so, if z lies in some compact subset K of $\mathbb{H}$, we obtain uniform convergence of (4) on K providing that the series $\sum |c|^{-2t}$ converges.

The convergence of the series $\sum |c|^{-2t}$ for sufficiently large values of t is easily established. For example, as $|g'(z)|^2$ represents the distortion in the *Euclidean* area of the mapping g, we can take a small disc D in $\mathbb{H}$ in a fundamental domain for G and then the Euclidean area of $g(D)$ is approximately $|g'(z)|^2\mathcal{A}$, where $\mathcal{A}$ is the Euclidean area of D. As the sets $g(D)$, $g \in G$, are disjoint sets contained in $\mathbb{H}$ (of finite Euclidean area π), the series $\sum |c|^{-4}$ converges. This argument can be made precise without difficulty, but it is natural to object to it on the grounds that it invokes Euclidean arguments. It can be modified to use hyperbolic arguments (although one should note that $\mathbb{H}$ has infinite hyperbolic area) and in this way one can show that $\sum |c|^{-2t}$ converges for all $t > 1$. In fact, this is the best one can do, for there are groups for which the series diverges when $t = 1$ (the groups whose fundamental region has finite area, and which correspond to compact Riemann surfaces

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with finitely many punctures). For a given group G, one can examine the exponent of convergence of the series for that group (essentially, the smallest t for which the series (4) converges); for most groups this is the Hausdorff dimension of the limit set (the accumulation points of the orbits of G) and it is an important measure of the (common) rate at which the G-orbits move out to the ideal boundary of $\mathbb{H}$.

4. The geometry of Riemann surfaces

Before we turn our attention to cover groups of hyperbolic Riemann surfaces, let us briefly consider the ideas involved by examining the effect of discreteness on Euclidean groups. Suppose that Γ is a discrete group of Euclidean isometries acting on the Euclidean plane, and that Γ contains translations. By discreteness, the translations in Γ have a minimal translation length which we may assume is 1 and attained by the translation t in Γ.

Let g in Γ be a rotation of angle $2\pi/n$ about the point z_g, so that the conjugate element $h = tgt^{-1}$ is a rotation of the same angle about $t(z_g)$. Now draw a line L_2 through z_g and $t(z_g)$, a line L_1 through z_g making an angle π/n with L_2, and a line L_3 parallel to L_1 through $t(z_g)$. Let α_j denote reflection in L_j. Then g (or g^{-1}) is $\alpha_1\alpha_2$, h^{-1} (or h) is $\alpha_2\alpha_3$, and gh (or a similiar word) is the translation $\alpha_1\alpha_3$ through a distance $2|z_g - t(z_g)|\sin(\pi/n)$. As this is at least 1, and as $|z_g - t(z_g)| = 1$, we conclude that $n \leq 6$.

This is the familiar crystallographic restriction, but the essential point here is to realise that these techniques are available in other geometries although the conclusions are different. The conclusions are different, of course, because each geometry comes equipped with its own trigonometry and the quantitative results of this type must, of necessity, reflect that particular trigonometry. To see the effect that the different trigonometries have on the three geometries, observe that Pythagoras' Theorem for a right angled triangle with sides a and b and hypothenuse c is $a^2 + b^2 = c^2$ in the Euclidean plane, whereas in the hyperbolic plane it is

$$\cosh a \cosh b = \cosh c,$$

and in the spherical case,

$$\cos a \cos b = \cos c.$$

It is only in the spherical case that we can have $a = b = c$. Note also that in the hyperbolic plane, for very large triangles we have (essentially) $a + b = c + \log 2$, that is, the vertices appear to be almost collinear (this is the effect of negative curvature). Again, in the hyperbolic plane there are regular n-gons with all angles $\pi/2$ precisely when $n \geq 5$; the case $n = 4$ is Euclidean and $n = 3$ is spherical.

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In the hyperbolic plane the circumference of a circle grows exponentially with the radius, and roughly at the same rate as the area, so that we may regard the hyperbolic plane as having, by comparison with the Euclidean plane, an immense amount of room near its ideal boundary (the circle at infinity): formally, this is the effect of negative curvature and separating geodesics and *it is this that accounts for the inexhaustible supply of discrete hyperbolic groups compared with the* 17 *wallpaper groups in the Euclidean case.* Roughly speaking, we can construct an increasing sequence of discrete hyperbolic groups in which in each case there is always ample room in the space near infinity to incorporate some extra group action into the picture.

In order to understand the mechanism by which discreteness imposes geometric constraints on hyperbolic Riemann surfaces, we need to know that an individual hyperbolic isometry g can be expressed as the composition $g = \alpha\beta$ of two reflections α and β across hyperbolic geodesics ℓ_α and ℓ_β, respectively (called the *axes* of α and β). We omit the proof of this, but remark that it is merely the hyperbolic counterpart of the familiar Euclidean fact that the composition of two reflections is either a rotation (if the axes of the reflection meet) or a translation (if the axes are parallel). It is a consequence of the failure of the Parallel Axiom in hyperbolic geometry that there are *three* possibilities in the hyperbolic plane, for there two geodesics either cross or not and, if not, they may or may not meet on the circle at infinity.

If ℓ_α and ℓ_β have a common orthogonal geodesic L_g; then g leaves L_g invariant. In fact, one end-point of L_g is an attracting fixed point, the other end-point is a repelling fixed point, and g moves each point of L_g by the same distance (twice the hyperbolic distance between ℓ_α and ℓ_β) along L_g. We call L_g the *axis* of g, the distance g moves each point along L_g is the *translation length* T_g of g, and g is said to be a *hyperbolic translation* (or, sometimes, a *loxodromic* isometry or even, rather confusingly, a hyperbolic element of the isometry group). In any event, if g lies in some cover group, the lines ℓ_α and ℓ_β cannot cross (else g would fix the point of intersection) and the only other case is that the two lines meet on the circle at infinity; in this case, g is said to be *parabolic* and it is conjugate to a Euclidean translation. In the case of a hyperbolic translation g, there is a useful formula for the distance a point z is moved by g, namely

$$\sinh \tfrac{1}{2}\rho(z, gz) = \sinh(\tfrac{1}{2}T_g) \cosh \rho(z, L_g)\,;$$

thus the minimum movement occurs on the axis L_g of g, and the further z is away from the axis, the more it is moved by g.

Now consider two hyperbolic translations g and h such that the axes L_g and L_h have a common orthogonal geodesic ℓ_β (and are therefore necessarily disjoint). It is easy to see that these can be expressed in the form $g = \alpha\beta$ and $h = \beta\gamma$, where α, β and γ are reflections with axes ℓ_α, ℓ_β and ℓ_γ, respectively, and, as

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a consequence of this, we have $gh = \alpha\gamma$; in other words, the composition for gh is expressed neatly in terms of the compositions for g and h. Sadly, this argument is restricted to two dimensions for in higher dimensions g and h may each be a composition of more that two reflections and then cancellation (of β, for example) may not occur.

We now examine some of the consequences of this expression for gh. If L_g and L_h come close to each other compared with T_g and T_h, and if the lines ℓ_α and ℓ_γ are on the same side of ℓ_β (which may be achieved by replacing g by g^{-1} if necessary), then the lines ℓ_α and ℓ_γ will meet and gh will have a fixed point in $\mathbb{H}$. The reader is urged to draw a diagram to illustrate this but, roughly speaking, the lines L_g and L_h curve (in Euclidean terms) away from each other, and this forces the lines ℓ_α and ℓ_γ to cross providing that T_g and T_h are not too large. If g and h are in some cover group then gh cannot have fixed points in $\mathbb{H}$ and so, in any cover group, the geometric quantities T_g, T_h and $\rho(L_g, L_h)$ (that is, the hyperbolic distance between the axes of g and h) cannot all be small. As the axes of g and h project to closed geodesic loops on the Riemann surface $\mathcal{R}$, this result tells us that *two short disjoint geodesic loops on $\mathcal{R}$ must be fairly far apart* or, equivalently, if they are close to each other then one is fairly long. The same argument applies when g and h are in the same conjugacy class in G, and this implies that if the distance from a geodesic loop to itself along a non-trivial closed curve on $\mathcal{R}$ is small, then the loop must be long. There are many results of this type available, and they all have a precise, quantitative, formulation which can be derived from elementary hyperbolic trigonometry; for example, the result above is that, in all cases,

$$2\sinh(\tfrac{1}{2}T_g)\sinh(\tfrac{1}{2}T_h)\sinh^2\tfrac{1}{2}\rho(L_g, L_h) \geq 1. \tag{5}$$

Roughly speaking, these results describe the universal metric properties of handles on a hyperbolic Riemann surface.

There is an analogous inequality to (5), valid when the two axes cross at an angle θ, and this is

$$\sinh(\tfrac{1}{2}T_g)\sinh(\tfrac{1}{2}T_h)|\sin\theta| \geq 1. \tag{6}$$

Suppose now that σ is a self-intersecting loop on a Riemann surface $\mathcal{R}$ of length $|\sigma|$. Then, in terms of the cover group G, this means that there is a hyperbolic translation g and a conjugate element hgh^{-1} such that both have translation length $|\sigma|$, and such that their axes cross at some angle θ. Applying (6), and using the fact that $1 \geq |\sin\theta|$, we see that $\sinh(\frac{1}{2}|\sigma|) \geq 1$; thus there is a lower bound on the length of a non-simple loop on a hyperbolic Riemann surface, and this is a universal lower bound in the sense that *it does not depend on the particular surface.*

A related result, quite beautiful in its simplicity, is that if G is a group of hyperbolic isometries without hyperbolic rotations, then, for all z in $\mathbb{H}$,

$$\sinh\tfrac{1}{2}\rho(z, gz)\sinh\tfrac{1}{2}\rho(z, hz) \geq 1 \tag{7}$$

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unless g and h lie in some cyclic subgroup of G (see [2]). In this result, G is *not* assumed to be discrete, and it is a consequence of this that *any group of hyperbolic motions without rotations is automatically discrete*, and hence automatically a cover group of some surface. Equally, if G is a cover group, this result is still applicable and it implies (but in a rather stronger form) that the lengths of any two loops from any point in the surface $\mathcal{R}$ can only both be small when they lie in a cyclic subgroup of the homotopy group. Again, this is a quantitative universal result which holds for all hyperbolic Riemann surfaces. Of course, (7) need not hold if $\langle g, h\rangle$ is cyclic; for example, one can construct a small translation g and then put $h = g^2$.

Results of this type are not restricted to hyperbolic translations, and there are similiar conclusions to be drawn for all elements in a cover group G. Briefly, a puncture p on the surface $\mathcal{R}$ corresponds to the unique fixed point ζ of a parabolic element g of G and, conversely, any parabolic element determines a puncture on $\mathcal{R}$ (we recall that a parabolic element is conjugate to the Euclidean translation $z \mapsto z + 1$ acting on the upper half-plane model of $\mathbb{H}$).

It can be shown that each such parabolic fixed point ζ is the point of tangency of a horocycle cal H_g in $\mathbb{H}$ (cal H_g is a Euclidean disc in $\mathbb{H}$ tangent to $\{z : |z| = 1\}$ at ζ) and, moreover, the horocycles cal H_g can be chosen to be disjoint for distinct ζ and to be compatible with conjugation in the sense that $h(\text{cal}\, H_g) = \text{cal}\, H_{hgh^{-1}}$. In fact, the elements of G leaving cal H_g invariant form a cyclic subgroup G_0 of G and the quotient space cal H_g/G_0 is a once punctured disc which is conformally equivalent to a neighbourhood of the corresponding puncture p on $\mathcal{R}$. This is the formal statement of the fact that the neighbourhood of any puncture on any hyperbolic Riemann surface is the same as the quotient of $\{x + iy : y > 0\}$ by the exponential map. As far as geometric constraints are concerned, one can show that the cal H_g can be chosen so that the (finite) area of cal H_g/G_0 is at least 1 (a universal lower bound), and that in an appropriate and universal sense the simple geodesic loops on the surface $\mathcal{R}$ do not come too close to the puncture p. The geometry near a puncture p is such that one travels an infinite distance to reach p, but through a neighbourhood of p of finite area; thus, roughly speaking, the surface $\mathcal{R}$ has an infinitely tall, infinitely thin, spike at p.

It is natural to now allow our groups to contain hyperbolic rotations and so study the most general discrete group G acting on the hyperbolic plane: this amounts to considering branched coverings of Riemann surfaces. In fact, if G is finitely generated, then it contains a torsion-free normal subgroup of finite index (this is *Selberg's Lemma*) and so we may expect that suitably, but only slightly, relaxed versions of the earlier constraints will hold. This is so but, of course, there are also entirely new types of constraints to be considered, namely those involving rotations.

Briefly, we describe some of the many geometric constraints that hold for these

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