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Hyperbolic Geometry

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INTRODUCTION

What is hyperbolic geometry? Let me try to give an answer by telling the story of the parallel axiom. I shall use modern language which will ruin part of the story but highlight the basic points.

Axioms for plane geometry A simple set of axioms for plane geometry can be presented in the framework of metric spaces. By a line in a metric space X we understand the image of a distance preserving map $\gamma:\mathbb{R}\to X$. The three axioms of plane geometry are (the axioms are analysed in an appendix)

INCIDENCE AXIOM Through two distinct points of X there passes a unique line. The space X has at least one point.

REFLECTION AXIOM The complement of a given line in X has two connected components. There exists an isometry σ of X which fixes the points of the line, but interchanges the two connected components of its complement.

PARALLEL AXIOM Through a given point outside a given line there passes a unique line which does not intersect the given line.

Investigations of the parallel axiom by among others J.Bolyai (1802 – 1860), C.F.Gauss (1777 – 1855), N.I.Lobachevsky (1793 – 1856) show that this axiom is independent of the other axioms in the sense that there exists a plane, the so called hyperbolic plane H^2, which satisfies the first two axioms of plane geometry but not the parallel axiom. H^2 is unique in the following sense.

CLASSIFICATION THEOREM A metric space which satisfies the three axioms of plane geometry is isometric to the Euclidean plane. A space which satisfies the first two axioms but not the parallel axiom is isometric to H^2 (after rescaling[1]).

The discovery of the hyperbolic plane began with a series of attempts to prove the parallel axiom from the other axioms. Such attempts should inevitably lead to the discovery of the hyperbolic plane since a system of axioms for the hyperbolic plane consists in the incidence axiom, the reflection axiom and the negation of the parallel axiom. G.Sacheri (1667 – 1733) was the first to penetrate deeper into this universe. He was followed by J.H.Lambert (1728 – 77) who made some important speculations on the nature of a non-Euclidean geometry. By 1850 most of the properties of a new geometry were known to Bolyai, Gauss and Lobachevsky. At that time the biggest problem was the existence of such a geometry! Twenty years later this was firmly settled through the works of E.Beltrami (1835 – 1900), A.Cayley (1821 – 95) and F.Klein (1849 – 1925). For more information on the history of hyperbolic geometry see [Milnor] or [Greenberg] and the references given there.

The Poincaré half-plane In 1882 H.Poincaré (1854 – 1912) discovered a new model of H^2 , which I shall now describe. The model is the upper half-plane of the complex plane equipped with the metric given by

$$\cosh d(z,w) = 1 + \tfrac{1}{2}|z-w|^2\, Im[z]^{-1}\, Im[w]^{-1}$$

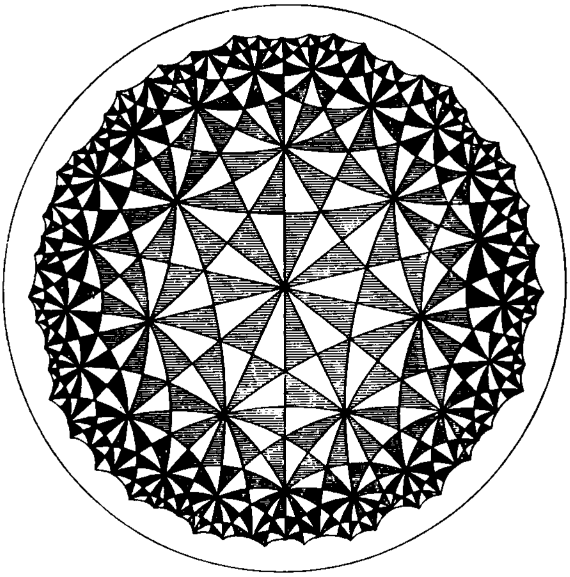
The lines in the Poincaré half-plane are traced by Euclidean circles with centres on the x–axis or Euclidean lines perpendicular to the x–axis. It is quite obvious that this model satisfies the incidence and reflection axioms but not the parallel axiom: Euclidean inversions furnish the isometries required by the reflection axiom.

The usefulness of the Poincaré half-plane model lies in the fact that the group of orientation preserving isometries consists of analytic transformations of the form

$$z \mapsto \frac{az+b}{cz+d} \qquad ; \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sl_2(\mathbb{R})$$

In fact the full group of isometries of the Poincaré half-plane is the group $PGl_2(\mathbb{R})$. In particular the group $Gl_2(\mathbb{Z})$ acts as isometries on H^2, a fact which lies at the root of the applications of hyperbolic geometry to number theory.

The Poincaré disc The open unit disc D is the second most commonly used model for the hyperbolic plane. We shall not write down the metric here, but mention that the lines in this model are traced by Euclidean circles and lines orthogonal to the unit circle ∂D. The Poincaré disc has certain aesthetic qualities, as shown below by a one of the many beautiful illustrations from [Klein, Fricke].



Discrete groups Reflections in the of lines of the tesselation above generate a discrete group of isometries of H^2. The systematic study of such groups was initiated by Poincaré in his "Memoir sur les groupes fuchsienne" 1882. Starting from a discrete group Γ of isometries of H^2 he used a procedure due to Dirichlet to construct a polygon Δ with a side pairing, i.e. instructions for identification of the sides of the polygon, which allows us to reconstruct the space H^2/Γ. Then he took up the converse problem: given a hyperbolic polygon with a side pairing, when does the side pairing generate a discrete group with the given polygon as fundamental domain. Poincaré laid down necessary and sufficient conditions for a given polygon with side pairings to determine a discrete group. The theorem has a bonus: from the geometry of the polygon we can read off a complete system of generators and relations for the group. In this way the topic has had a big influence on what today is known as combinatorial group theory. The theorem of Poincaré is one of the main themes of this book. The proof of the theorem is based on some geometric ideas which I shall outline next.

New geometries Let us for a moment go back to the discussion of the parallel axiom and recall that the hyperbolic plane is quite unique. However, we find numerous new geometries in the class of hyperbolic surfaces: spaces locally isometric to H^2. We shall see that complete hyperbolic surfaces can be classified by discrete subgroups of $PGl_2(\mathbb{R})$. The main point in the proof of Poincaré's theorem is the monodromy theorem: A local isometry $f{:}X \to H^2$ from a complete hyperbolic surface X to the hyperbolic plane H^2 is an isomorphism.

Higher dimensions So far we have only talked about hyperbolic geometry in two dimensions, but the hyperbolic space H^n exists in all dimensions. It is the second main theme of this book to construct these spaces and to identify their isometry groups with the Lorentz group known from special relativity. Hyperbolic n-space is constructed as one of the two sheets of the hyperbola

$$x_0^2 - x_1^2 - x_3^2 - x_4^2 ..... - x_n^2 = 1$$

It turns out that the most natural framework for this construction is the theory of quadratic forms. This is actually where the books starts. We offer a general introduction to quadratic forms in Chapter I as an opener to the construction of geometries in Chapter II. It is natural to develop Euclidean and spherical geometries in a parallel fashion in order to be able to draw on analogies with these more familiar geometries. A special feature of hyperbolic geometry is the presence of a natural boundary ∂H^n. In terms of the hyperbola above, a boundary point can be thought of as an asymptote to the hyperbola, i.e. line in the cone

$$x_0^2 - x_1^2 - x_3^2 - x_4^2 ..... - x_n^2 = 0$$

It follows that the boundary ∂H^n is a sphere. In the end we find that the Lorentz group acts on the sphere as the group of Möbius transformations.

Three-dimensional geometry A specific feature of three-dimensional hyperbolic geometry is that the group $Sl_2(\mathbb{C})$ operates on H^3 (see below) in such a way as to give an isomorphism between $PGl_2(\mathbb{C})$ and the group of orientation preserving isometries of H^3, the special Lorentz group. This has the consequence that a Kleinian group, i.e. is a discrete subgroup Γ of $Sl_2(\mathbb{C})$, defines a 3-manifold H^3/Γ. The work of W.P.Thurston on 3-manifolds [Thurston] has generated new interest in hyperbolic geometry.

Exterior algebra Let us think of H^3 as a sheet of the unit hyperbola in a Minkowski space M, i.e. a four-dimensional vector space with a form of type (−3,1). Once we pick an orientation of M, the space $\wedge^2 M$ comes equipped with the structure of a three-dimensional complex vector space with a complex quadratic form Q. The assignment of normal vector $\boldsymbol{n} \in \wedge^2 M$ to an oriented geodesic line h in H^3 creates a bijection between the complex quadric $Q = -1$ and the set of oriented geodesics in H^3.

The Hermitian model As mentioned above, each Minkowski space M comes equipped with its own copy of H^3, a sheet of the unit hyperbola. It turns out that there exist Minkowski spaces with more algebraic structures, in particular a structure of a linear representation of $Sl_2(\mathbb{C})$. We shall be interested in the space M of 2×2 Hermitian matrices, i.e. complex matrices of the form

$$\begin{bmatrix} a & b \\ \bar{b} & d \end{bmatrix} \qquad ; a, d \in \mathbb{R}, b \in \mathbb{C}$$

The restriction of the determinant to M is a quadratic form of type (−3,1). The sheet of the unit hyperbola ($ad - b\bar{b} = 1$) consisting of positive definite forms ($a > 0$) is called the Hermitian model of H^3. The action of $Sl_2(\mathbb{C})$ on M is given by the formula

$$\sigma X = \sigma X \sigma^* \qquad ; \sigma \in Sl_2(\mathbb{C}),\ X \in M$$

The key fact about the Hermitian model is the following trace formula: The action of $S \in Gl_2(\mathbb{C})$ on H^3 can be decomposed into $\alpha\beta$ where α and β are half-turns with respect to geodesics a and b. For any such decomposition we have that

$$tr^2(S) = 4 < \boldsymbol{m}, \boldsymbol{n} >^2$$

where $\boldsymbol{m}$ and $\boldsymbol{n}$ denote normal vectors for the geodesics a and b.

The realisation of Minkowski space through the space M of Hermitian matrices has been used in physics for a long time. We shall introduce another tool from physics namely the Dirac algebra, which is a concrete realisation of the Clifford algebra of M. The use of Clifford algebras was suggested by the "strange formulas" from the book of [Fenchel].

Prerequisites The general prerequisites are linear algebra and a modest amount of point set topology, including compact subsets of a finite dimensional real vector space. Familiarity with the concepts of group theory and, for the understanding of Poincaré's polygon theorem, a bare minimum of combinatorial group theory ("generators and relations") is needed. Elementary topology is needed for VI.8, VII.5, VII.7, while VII.6 is a good introduction to combinatorial topology. The final chapter of the book requires familiarity with the exterior algebra.

University of Aarhus, Denmark. July 1992 Birger Iversen

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