I QUADRATIC FORMS

In the first two sections we lay down the general principles for quadratic forms and the associated orthogonal groups. The third section will classify real quadratic forms into Sylvester types, while the remaining sections will deal with real forms of specific types.

Three types of real quadratic forms are of particular interest: positive definite forms, parabolic forms and hyperbolic forms. The corresponding orthogonal groups provide us with the isometry groups of the three basis geometries: spherical geometry, Euclidean geometry and hyperbolic geometry.

For example, the orthogonal group of a hyperbolic form contains the Lorentz group as a subgroup of index 2. The Lorentz group, on the one hand, is the isometry group of hyperbolic geometry, while, on the other hand, it can be identified with the somewhat older group of Möbius transformations. We shall see that "the Möbius group is the boundary action of the Lorentz group".

I.1 ORTHOGONALITY

Let k denote a field[1] of characteristic $\neq 2$ and E a vector space over k of finite dimension n. By a quadratic form on E we understand a function Q: E→k homogeneous of degree 2, i.e.

1.1 $$Q(\lambda z) = \lambda^2 Q(z) \qquad ; \lambda \in k,\ z \in E$$

with the property that the symbol

1.2 $$<x,y> = \tfrac{1}{2}(Q(x+y) - Q(x) - Q(y)) \qquad ; x,y \in E$$

[1] In this text we have applications only for the field $\mathbb{R}$ of real numbers, the field $\mathbb{C}$ of complex numbers and occasionally the field $\mathbb{Q}$ of rational numbers.

is bilinear in x and y. The relation 1.2 between the bilinear form $<x,y>$ and the quadratic form Q(z) is called the formula for polarization. Observe that the bilinear form $<x,y>$ is symmetrical in x and y.

Proposition 1.3 Polarization gives a one to one correspondence between quadratic forms and symmetrical bilinear forms over a field k with $char(k) \neq 2$.

Proof A quadratic form Q(z) satisfies the formula $Q(2z) = 4Q(z)$ as a special case of 1.1. We can now substitute z for x and y in formula 1.2 to get that

1.4 $$Q(z) = <z,z> \qquad ; z \in E$$

which recovers the function Q(z) from the symbol $<x,y>$. Conversely, if we start with a symmetrical bilinear form $<x,y>$ we can use formula 1.4 to define a k-valued function Q on E which is homogeneous of degree 2. Bilinearity and symmetry give us the formula

$$<x+y, x+y> = <x,x> + <y,y> + 2<x,y> \qquad ; x,y \in E$$

which shows that $Q(z) = <z,z>$ is a quadratic form. □

The evaluation, Q(x), of the quadratic form at a vector $x \in E$ is often referred to as the norm of x. Vectors $x,y \in E$ are called orthogonal if $<x,y> = 0$. Linear subspaces U and V of E are called orthogonal if all vectors in U are orthogonal to all vectors in V. For a given linear subspace K of E we define the total orthogonal subspace $K^\perp$ by the formula

1.5 $$K^\perp = \{ e \in E \mid \forall x \in K \ <e,x> = 0 \}$$

Definition 1.6 A quadratic form Q(z) on the finite dimensional vector space E over k is called non-singular if the bilinear form $<x,y>$ satisfies

$$\forall x \in E\ (\langle x,y\rangle = 0) \ \Rightarrow\ y = 0 \qquad ; y \in E$$

Otherwise expressed, the form $Q(z)$ is non-singular if and only if $E^{\perp} = 0$.

Theorem 1.7 Let E be a finite dimensional vector space equipped with a non-singular quadratic form Q. For any linear subspace K of E we have

$$\mathit{dim}\ K + \mathit{dim}\ K^{\perp} = \mathit{dim}\ E$$

Proof Let us recall that a linear form on E is nothing but a k–linear map $\xi: E \to k$. Linear forms may be added and multiplied by a constant from k to form a new vector space, the dual space E^*. Let us prove that

1.8 $$\mathit{dim}\ E^* = \mathit{dim}\ E$$

To this end we simply pick a basis $e_1,\dots,e_n$ for E and introduce the coordinate forms $x_1,\dots,x_n \in E^*$ by the formula

$$e = \sum\nolimits_i x_i(e)\, e_i \qquad ; e \in E$$

Let us verify that any linear form ξ on E satisfies the formula

$$\xi = \sum\nolimits_i \xi(e_i)\, x_i \qquad ; \xi \in E^*$$

To this end we simply evaluate both sides of the formula on the basis $e_1,\dots,e_n$. This shows that $x_1,\dots,x_n$ generates the vector space E^*. To see that $x_1,\dots,x_n$ are linearly independent, consider a relation of the form $\sum \lambda_i x_i = 0$, $\lambda_1,\dots,\lambda_n \in k$. Evaluate the relation on e_j, $j = 1,\dots,n$, and conclude that $\lambda_j = 0$.

Let us introduce the map

1.9 $$q: E \to E^* ,\ \ q(e) = (\ x \mapsto \langle e,x\rangle\ ) \qquad ; e \in E$$

Observe that the kernel for q is $E^{\perp}$ and use that Q is non-singular to conclude that q is injective; formula 1.8 allows us to conclude that q is surjective as well. Restriction of a form along the inclusion $i: K \to E$ defines a k–linear map

$$i^*: E^* \to K^* \qquad ; \xi \mapsto \xi \circ i,\ \xi \in E^*$$

which is surjective: In the argument above pick the basis $e_1,\dots,e_n$ for E such that $e_1,\dots,e_k$ is a basis for K and observe that the coordinate forms for this basis are $x_1 \circ i,\dots,x_k \circ i$. We shall focus on the composite map

$$E \to^{q} E^* \to^{i^*} K^*$$

Since this is surjective we conclude from the dimension formula of Grassmann[2]

$$\mathit{dim}\ E = \mathit{dim}\ K^* + \mathit{dim}\ \mathit{Ker}\ i^*q$$

We ask the reader to show that $\mathit{Ker}\ i^*q = K^{\perp}$. The result follows from the formula above in combination with 1.8. □

Let us consider a pair (E,Q) consisting of a finite dimensional vector space E and a quadratic form Q on E. By an isomorphism from (E,Q) to a second such pair (F,P) we understand a linear isomorphism $\sigma: E \tilde{\to} F$ such that $Q = P \circ \sigma$. In particular we may talk about automorphisms of (E,Q) which may be composed to form a group, the orthogonal group of (E,Q) which is denoted $O(Q)$ or just $O(E)$ when no confusion is possible.

Let us quite generally consider a quadratic form (E,Q) and pick a basis $e_1,\dots,e_n$ for the vector space E. The Gram matrix $G \in M_n(k)$ is given by

1.10 $$G_{ij} = \langle e_i, e_j \rangle \qquad ; i,j = 1,\dots,n$$

Proposition 1.11 Let Q be a non-singular quadratic form on E. An orthogonal transformation $\sigma \in O(Q)$ has determinant 1 or -1.

Proof Let us pick a basis $e_1,\dots,e_n$ for E and let A denote the matrix for σ, thus $\sigma(e_j) = \sum_i A_{ij} e_i$ for $j = 1,\dots,n$. Direct calculation gives us

$$\langle \sigma(e_i), \sigma(e_j) \rangle = \langle \textstyle\sum_k A_{ki} e_k, \sum_h A_{hj} e_h \rangle = \sum_{h,k} {}^{T}\!A_{ik} \langle e_k, e_h \rangle A_{hj}$$

which shows that the Gram matrix for $Q \circ \sigma$ is ${}^{T}\!AGA$. In particular we find that σ is an orthogonal transformation for Q if and only if the matrix A obeys

[2] A linear map $f: E \to F$ between finite dimensional vector spaces satisfies

$$\mathit{dim}\ E = \mathit{dim}\ \mathit{Im}(f) + \mathit{dim}\ \mathit{Ker}(f)$$

1.12
$$^{T}A\ G\ A = G$$

Let us observe that the Gram matrix G is the matrix for the map $q:E \to E^*$ with respect to the basis $e_1,\dots,e_n$ for E and $x_1,\dots,x_n$ for E^*. Since Q is non-singular we conclude that $det\ G \neq 0$. The formula 1.12 gives us

$$det\ ^{T}A\ det\ G\ det\ A = det\ G$$

from which we conclude that $det^2 A = 1$, or $det\ A = 1$ or -1. □

The orthogonal group for the standard quadratic form on k^n

1.13
$$Q(x) = x_1^{\,2}+x_2^{\,2}+ \dots + x_n^{\,2} \qquad ; x = (x_1,\dots,x_n) \in k^n$$

is denoted $O_n(k)$. The subgroup of transformations with determinant 1 is denoted $SO_n(k)$. When $k = \mathbb{C}$ or any other algebraically closed field with $char(k) \neq 2$ any non-singular quadratic form is isomorphic to the standard form 1.13.

Theorem 1.14 Let Q be a non-singular quadratic form on the vector space E over $\mathbb{C}$ of dimension n. There exists a basis $e_1,\dots,e_n$ for E with

$$<e_i,e_j> = \delta_{ij} \qquad ; i,j = 1,\dots,n$$

Proof Let us observe that Q can't be identically zero since this implies that the bilinear form $<x,y>$ is identically zero. Thus we can find a vector $v \in E$ with $Q(v) \neq 0$. Let us write $Q(v) = \lambda^2$, $\lambda \in \mathbb{C}$, or $Q(\lambda^{-1}v) = 1$ to conclude that we can find $e \in E$ with $Q(e) = 1$. Let K denote the line spanned by e and put $F = K^{\perp}$ and observe that $E = K \oplus F$ (meaning that any vector $x \in E$ has a unique decomposition $x = f + k$, where $f \in F$ and $k \in K$). It is easily seen that the restriction of Q to F is non-singular. It is left to the reader to complete the proof by simple induction on $dim(E)$. □

I.2 WITT'S THEOREM

In this section we shall be concerned with a non-singular quadratic form Q on a vector space E of dimension n over a field k with $char(k) \neq 2$. The theme is the flexibility of the orthogonal group $O(Q)$. The tool is reflections, an important class of orthogonal transformations, which we proceed to introduce.

We say that a vector $n \in E$ is non-isotropic if $Q(n) \neq 0$. A non-isotropic vector $n \in E$ defines a linear transformation τ_n of E by the formula

2.1
$$\tau_n(x) = x - 2\,\frac{<x,n>}{<n,n>}\,n \qquad ; x \in E$$

Direct computation gives us

$$<\tau_n(x),\tau_n(x)> = <x,x> \qquad ; x \in E$$

i.e. τ_n is an orthogonal transformation called reflection along n. The hyperplane orthogonal to n is fixed by τ_n while $\tau_n(n) = -n$. It follows that τ_n is an involution, i.e. $\tau_n{}^2 = \iota$ but $\tau_n \neq \iota$, and that

2.2
$$det\ \tau_n = -1 \qquad ; n \in E,\ <n,n> \neq 0$$

Proposition 2.3 Let Q be a quadratic form on the vector space E. For any $\lambda \in k^*$, the orthogonal group $O(Q)$ acts transitively[3] on the set

$$\{\, e \in E \mid Q(e) = \lambda \,\}$$

Proof Consider vectors $e,f \in E$ with $Q(e) = Q(f) = \lambda \neq 0$. Observe, that

$$<e-f,\ e+f> = <e,e> - <f,f> = 0$$

It follows that we can't have $Q(e+f) = 0$ and $Q(e-f) = 0$ at the same time (the non-isotropic vector e is a linear combination of $e-f$ and $e+f$). If the vector $e-f$ is non-isotropic, reflection τ along this vector fixes $e+f$. Thus

$$\tau(e-f) = f-e \quad , \quad \tau(e+f) = e+f$$

[3] We say that the action of a group G on a set X is transitive, if for all $x \in X$ and $y \in X$ there exists $\sigma \in G$ with $\sigma(x) = y$.

Cambridge University Press
978-0-521-43528-4 - Hyperbolic Geometry
Birger Iversen
Excerpt
More information

Now, add these two formulas to see that $\tau(e) = f$. If $e + f$ is isotropic, reflection ρ along this vector interchanges e and $-f$. In conclusion the orthogonal transformation $\tau_f\rho$ maps e to f as required. □

When Q is non-singular, it is also true that $O(Q)$ acts transitively on the set of <u>isotropic</u> <u>lines</u>, i.e. lines in E generated by isotropic vectors. In fact,

Witt's theorem 2.4 Let (E,Q) be a non-singular quadratic form. Any isomorphism $\sigma\colon (U,Q) \tilde{\to} (V,Q)$, where U and V are linear subspaces of E, can be extended to an orthogonal transformation of (E,Q).

Proof Let us first treat the case where U is non-singular. This will be done by induction on $dim(U)$. In case $dim(U) = 1$ let us pick a non-zero $u \in U$ and put $\lambda = Q(u)$. Since $Q(\sigma(u)) = \lambda$, we can apply 2.3 to extend σ to an orthogonal transformation of E.

To accomplish the induction step let us pick a non-isotropic vector $e \in U$. Since $Q(e) = Q(\sigma(e))$ we can use 2.3 to pick $\tau \in O(E)$ such that $\sigma(e) = \tau(e)$. It will suffice to extend $\tau^{-1}\sigma\colon U \to E$ to an orthogonal transformation of E. To recapitulate it suffices to extend $\sigma\colon U \to E$ under the assumption that σ fixes a non-isotropic vector $e \in U$. Let us introduce the total orthogonal space W to $L = k.e$ in U and the total orthogonal space F to L in E. Observe that W and F are non-singular and that $W \subset F$ and $\sigma(W) \subset F$. Thus we can use the induction hypothesis to extend the restriction $\sigma\colon W \to F$ to an orthogonal transformation of F.

Let us now assume that U is singular. Let there be given data consisting of a non-zero vector $e \in U \cap U^{\perp}$ and a linear complement R to L in U. We are going to show that there exists an isotropic vector f in E which is orthogonal to R and such that $<e,f> = 1$. Observe that we can find $\beta \in E^*$ with $\beta(e) = 1$ and $\beta(R) = 0$; the vector $b \in E$ with $q(b) = \beta$ is orthogonal to R and satisfies $<b,e> = 1$. Let us show that the vector $f = b - \frac{1}{2}<b,b>e$ is isotropic

$$<f,f> = <b,b> - <b,b><e,b> = 0$$

The isotropic vector f is orthogonal to R and satisfies $<e,f> = 1$. Let us observe that $f \notin U$ since e is orthogonal to U but $<e,f> = 1$. The space $W = U + \mathbb{R}f$ is the direct sum of the two orthogonal subspaces R and $ke + kf$. Let us perform the same construction on the data $\sigma(U)$, $\sigma(e)$, $\sigma(R)$ to find an isotropic vector $g \in E$ which is orthogonal to $\sigma(R)$ and which satisfies $<g,\sigma(e)> = 1$. Let us extend $\sigma: U \to E$ to a linear map $W \to E$ by the convention $\sigma(f) = g$. It is easy to check that the extended map preserves the inner product: use that $\sigma(W)$ is the direct sum of the orthogonal subspaces $\sigma(R)$ and $k\sigma(e) + k\sigma(f)$.

Let us finally consider the general subspace U of E. If U is non-singular we can use the first part of the proof to extend σ. If U is singular we can use the middle part of the proof a number of times until U becomes non-singular, and then apply the first part of the proof. □

The orthogonal group $O(E)$ of a non-singular form is generated by reflections along non-isotropic vectors as it follows from exercise 2.3. The number of reflections needed can be bounded by the following theorem of Elie Cartan: An orthogonal transformation of a non-singular form of dimension n can be written as a product of no more that n reflections along non-isotropic vectors.

The proof of the theorem of E. Cartan is rather complicated, see [Berger] and [Deheuvels]. Moreover, the theorem is not precise enough for our purposes. For these reasons we shall not make use of the theorem in its generality, but prove a number of more specific results along these lines: namely 4.5, 5.3, 6.11

I.3 SYLVESTER TYPES

In this section we shall deal exclusively with quadratic forms over the field $\mathbb{R}$ of real numbers. By a Euclidean vector space we understand a finite dimensional vector space E with a quadratic form Q which is positive definite, i.e. $Q(e) > 0$ for all non-zero $e \in E$. The form Q is called negative definite if $Q(e) < 0$ for all non-zero $e \in E$. Our first objective is to generalise the concept of an orthonormal basis.

Proposition 3.1 Let (E,Q) be a quadratic form on the vector space E over $\mathbb{R}$ of dimension n. There exists an orthonormal basis for E, i.e. a basis $e_1,\dots,e_n$ with

$$<e_i,e_j> = \begin{cases} 0 & ; \ i \neq j \\ -1,\ 0,\ +1 & ; \ i = j \end{cases}$$

Proof Let us use induction with respect to $dim(E) = n$. If the form Q is identically 0, any basis for E is orthonormal. Thus we may assume the existence of an $e \in E$ with $Q(e) \neq 0$. Let us write $Q(e) = \epsilon\lambda^2$ with $\epsilon = \pm 1$ and $\lambda \in \mathbb{R}^*$. Put $e_1 = \lambda^{-1}e$ to get $Q(e_1) = \pm 1$. Let us use the induction hypothesis to find an orthonormal basis $e_2,\dots,e_n$ for $(\mathbb{R}e_1)^\perp$. The basis $e_1,\dots,e_n$ meets our requirements. □

Sylvester's theorem 3.2 Let $e_1,\dots,e_n$ be an orthonormal basis for the quadratic form (E,Q) over $\mathbb{R}$. The numbers

$$p = \#\{ i \mid <e_i,e_i> = -1 \}, \quad q = \#\{ i \mid <e_i,e_i> = 1 \}$$

are independent of the orthonormal basis considered.

Proof Let us fix the basis as in the statement of the proposition, and let E_- denote the subspace of E, generated by those elements e_i from our basis for which $<e_i,e_i> = 0, -1$. Observe that $Q(e) \leq 0$ for all $e \in E_-$ and conclude that for any Euclidean subspace F of E we have $F \cap E_- = 0$. According to 3.4 we have that

$$dim(F) + dim(E_-) = dim(F + E_-) + dim(F \cap E_-)$$

This gives us $dim(F) + n - q \leq n$, i.e. $dim(F) \leq q$. It follows that

3.3 $$sup \{ dim(F) \mid \text{Euclidean } F \subseteq E \} = q$$

This shows that q is independent of the basis chosen. Let us apply this to the space (E,−Q) and conclude that p too is independent of the basis. □

With the notation of the theorem we say that the quadratic form (E,Q) has Sylvester type (−p,q). Observe that two quadratic forms (E,Q) and (F,R) with $dim(E) = dim(F)$ of equal Sylvester type are isomorphic.

Dimension formula 3.4 For subspaces U and V of the finite dimensional vector space E we have that

$$dim(U \cap V) + dim(U+V) = dim(U) + dim(V)$$

Proof Let us apply the Grassmann dimension formula to the linear map

$$f : U \oplus V \to E, \quad f(u,v) = u - v \qquad ; u \in U, v \in V$$

observing that $Ker\, f \xrightarrow{\sim} U \cap V$ and $Im\, f = U + V$. □

Discriminant inequality 3.5 Let (E,Q) be a non-singular quadratic form of Sylvester type (−s,r). For any basis $e_1,\dots,e_n$ for E we have

$$\boxed{sign\ det_{ij} <e_i,e_j> \ = \ (-1)^s}$$

Proof Let us consider a second basis $f_1,\dots,f_n$ related to the first basis through the transition matrix B. This gives us

$$<f_i,f_j> = <\sum_r B_{ir}e_r, \sum_s B_{js}e_s> = \sum_{r,s} B_{ir}<e_r,e_s>{}^TB_{sj}$$

From this we conclude that

$$det_{ij} <f_i,f_j> = det\ B\ det_{rs}<e_r,e_s>\ det\ {}^TB = det^2B\ det_{ij}<e_i,e_j>$$