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Background

Throughout this treatise, Y (state space), U (control space), Z (observation space), and Z_f (final state observation space) are (separable) Hilbert spaces. Moreover, $A : Y \supset \mathcal{D}(A) \to Y$ is the infinitesimal generator of a strongly continuous (s.c.) semigroup e^{At}, $t \geq 0$ on Y.

Following the pioneering work of Kalman in 1960 in the finite-dimensional (matrix) case, the Linear Quadratic (LQ) optimal control problem, and related Riccati theory, for the dynamics

$$\dot{y} = Ay + Bu, \qquad y(0) = y_0 \in Y, \tag{0.1}$$

with quadratic cost functional

$$J(u, y) \equiv \int_0^T \left[\|Ry(t)\|_Z^2 + \|u(t)\|_U^2 \right] dt + \|Gy(T)\|_{Z_f}^2 \tag{0.2}$$

to be minimized over all $u \in L_2(0, T; U)$, has already received several treatments in book form, in the most amenable case where the control operator B, the observation operator R, and the final state operator G are all bounded:

$$B \in \mathcal{L}(U; Y); \quad R \in \mathcal{L}(Y; Z); \quad G \in \mathcal{L}(Y; Z_f). \tag{0.3}$$

In (0.2), T may be finite: $0 < T < \infty$, or else infinite: $T = \infty$, in which case we take $G = 0$. See, for example, J. L. Lions [1970], Curtain and Pritchard [1978], Balakrishnan [1981], Bensoussan et al. [1993], and Curtain and Zwart [1995] for a sample of more recent books. Thus, this preliminary topic, under assumptions (0.3), in particular, with *distributed* control (i.e., with $B \in \mathcal{L}(U; Y)$), is well covered in book form. Hence, we shall not pursue it here directly, but rather refer to any of the above references. The treatment in [Balakrishnan, 1981, Chapter 5] and the two-volume treatise [Bensoussan et al., 1993] are the most appropriate prerequisite for the present treatise, along with the Partial Differential Equations–oriented book [Lions, 1970]. Reference [Bensoussan et al., 1993] is also the most useful as a complementary work for topics such as periodic control, strict and classical solutions of the Riccati equations when

$B \in \mathcal{L}(U; Y)$, which are not touched upon in the present treatise, and also for its emphasis on Da Prato's *direct method* on the study of the Riccati equations, followed by the dynamic programming approach to solve the optimal control problem. In contrast, the present treatise will emphasize the more powerful, reverse approach, the variational approach, from the control problem to the Riccati equations pursued by the authors.

Moreover, references [Bensoussan et al., 1993] and [Lions, 1970] are, by far, the most mathematically advanced of the above group, as they set the focus on an in-depth, modern treatment of general PDEs (Partial Differential Equations), which are also the deliberate objective of the present treatise.

Instead, the massive recent reference [Curtain, Zwart, 1995] is useful on a different part of the spectrum, as an introductory textbook, which seeks to present a link between the finite-dimensional theory and the most amenable part of an "infinite-dimensional theory," that is, the one that deals with all bounded operators, except for the free dynamics operator. The corresponding theory is then illustrated by a wealth of detailed, much-appreciated examples, typically involving the basic heat and wave equations in one dimension with distributed control, as well as delay-differential equations.

By contrast, in the present treatise, we shall give emphasis throughout to the case where the control operator B is genuinely and "fully unbounded," such as it arises in the study of *boundary* or *point control* problems for general PDEs. While our approach is abstract, the setting is motivated by, and ultimately directed to, concrete classes of PDEs on a general multidimensional domain. Transition from the abstract to the concrete and conversely requires full familiarity with modern PDE theories and treatments. These will be assumed here as *prerequisites*. [The abstract treatment of the present books will naturally include, as a very special case, other differential dynamics, such as functional differential equations (hereditary or delay-equations), with delay in the state and/or control. However, we shall not explicitly focus on this functional differential subclass in our applications. As abstract systems, functional differential equations are far more amenable than any class of PDEs with boundary/point control. However, as delay-differential equations are defined in terms of matrices and "delay" constants, the final results need to be expressed in terms of these data. Thus, they deserve a special treatment of their own; see Manitius [1976] and [Bensoussan et al., 1993, Vol. I, Chapter 4] for two book-form expositions, along with [Curtain, Zwart, 1995, and many specialized articles.] Accordingly, in the present treatise, all the applications of the abstract theory – continuous as well as numerical theory – will refer to mixed (initial-boundary value) problems for PDEs. More precisely, the concrete PDE classes – parabolic, hyperbolic/Petrowski-type mixed problems – will motivate, and lead to, corresponding abstract mathematical settings in the chapters below.

In Volume I (Chapters 1 through 6), we shall consider the class of "abstract parabolic" problems. This is characterized by the property that the s.c. (free dynamics) semigroup e^{At} is *analytic* on Y, $t > 0$. Chapter 6 extends problem (0.1), (0.2) to a min–max game theory problem, with indefinite cost and to the indefinite cost optimal control case. Li and Yong [1995, Chapter 9] have recently studied the optimal control problem for abstract parabolic equations, with R^*R replaced by a nonnecessarily positive semidefinite operator, along with other mathematically advanced optimal control topics. This book [Li, Yong, 1995] also serves as an attractive companion to the present treatise, not least because Chapter 9 of [Li, Yong, 1995] on the LQ-problem adopts the abstract approach of the present authors of their original papers, which is the thrust of the present treatise.

In Volume II (with Chapters 7 through 10) pertaining to the optimal control problem for (0.1), (0.2) over a finite time interval, $T < \infty$, we shall consider mixed hyperbolic and Petrowski-type PDE problems as well.

0.1 Some Function Spaces Used in Chapter 1

To facilitate the reading of the statements of Theorems 1.2.1.1, 1.2.2.1, and 1.2.2.2 of Chapter 1, we shall list here for convenience a few function spaces used there.

If X is a Hilbert space and r any real number, then:

(i) ${}_rC([s,T];X)$ denotes the Banach space defined by

$$ {}_rC([s,T];X) \equiv \left\{ f(t) \in C((s,T];X) : \|f\|_{{}_rC([s,T];X)} = \sup_{s<t\le T} (t-s)^r \|f(t)\|_X < \infty \right\}; \tag{0.4} $$

(ii) $C_r([s,t];X)$ denotes the Banach space defined by

$$ C_r([s,T];X) \equiv \left\{ f(t) \in C([s,T);X) : \|f\|_{C_r([s,T];X)} = \sup_{s\le t<T} (T-t)^r \|f(t)\|_X < \infty \right\}. \tag{0.5} $$

(iii) If r_1 and r_2 are real numbers, then ${}_{r_1}C_{r_2}([0,T];X)$ denotes the Banach space defined by

$$ {}_{r_1}C_{r_2}([0,T];X) \equiv \left\{ f(t) \in C((0,T);X) : \|f\|_{{}_{r_1}C_{r_2}([0,T];X)} = \sup_{0<t<T} t^{r_1}(T-t)^{r_2} \|f(t)\|_X < \infty \right\}. \tag{0.6} $$

The above spaces measure, when $r,\ r_1,\ r_2 > 0$, the singularity of $f(t)$, as $t \to T$ or $t \to 0$.

0.2 Regularity of the Variation of Parameter Formula When e^{At} Is a s.c. Analytic Semigroup

Prerequisites to the present treatise include the *general theory of s.c. semigroups*, and operator theory in general, which is available in numerous excellent references, some of which are cited throughout this exposition, as needed. However, to facilitate the reading of the abstract parabolic Volume I, we shall collect here a few well-known results (which are scattered in the literature) that are often invoked in Volume I.

Proposition 0.1 *Let $A : Y \supset \mathcal{D}(A) \to Y$ be the infinitesimal generator of a s.c. analytic semigroup e^{At} on the Hilbert space Y, $t > 0$. Then, the Cauchy problem*

$$\dot{y} = Ay + f, \qquad y(0) = 0 \tag{0.7}$$

admits the following regularity properties, for any $0 < T < \infty$:

$$y(t) = (Lf)(t) = \int_0^t e^{A(t-\tau)} f(\tau)\, d\tau: \tag{0.8}$$

$$\text{(i) continuous } L_2(0,T;Y) \to L_2(0,T;\mathcal{D}(A)), \tag{0.9}$$

$$\text{(ii) continuous } L_2(0,T;Y) \to C\big([0,T];[\mathcal{D}(A),Y]_{\frac{1}{2}}\big), \tag{0.10}$$

$$\text{(iii) continuous } L_2(0,T;Y) \to C\big([0,T];\mathcal{D}\big((-A)^{\frac{1}{2}-\epsilon}\big)\big), \quad \forall \epsilon > 0, \tag{0.11}$$

$$\text{(iv) continuous } C([0,T];Y) \to C\big([0,T];\mathcal{D}\big((-A)^{1-\epsilon}\big)\big), \quad \forall \epsilon > 0 \tag{0.12}$$

[assuming, in (0.11) and (0.12), that the fractional powers are well-defined],

$$\text{(v) continuous } L_p(0,T;Y) \to L_p(0,T;\mathcal{D}(A)), \quad 1 < p < \infty, \tag{0.13}$$

generalizing (0.9).

Moreover, via (0.7), property (0.9) is equivalent to the following regularity properties:

$$f \to Ay, \dot{y} : \text{continuous } L_2(0,T;Y) \to L_2(0,T;Y). \tag{0.14}$$

0.2.1 Comments on the Space $[X,Y]_{\frac{1}{2}}$

In (0.10), $[X,Y]_{\frac{1}{2}}$ is the complex interpolation, or intermediate space, geometrically defined as in [Lions, Magenes, 1972, Vol. 1, Eqn. (2.7), p. 10], for X and Y Hilbert spaces, $X \subset Y$, X dense in Y with continuous injection, by means of domains of positive self-adjoint operations (or, equivalently, by the "complex interpolation method" as in [Lions, Magenes, 1972, Vol. 1, Section 14, pp. 91–94]). In fact, this definition is all that is needed in the Hilbertian setting of the present treatise. However, to embed this definition in the broader setting of interpolation theory, we recall that an equivalent definition, which is valid also in the Banach setting, is given in [Lions, Magenes, 1972, Vol. 1, Chapter 1, Section 15, p. 98] by the "real interpolation method" and leads to the spaces of averages $(X,Y)_{\theta,p}$ [Triebel, 1978]. When X and Y are Hilbert, and $p = 2$, then $[X,Y]_\theta = (X,Y)_{\theta,2}$ [Triebel, 1978, Remarks 3 and 4, p. 143]. A useful summary of this theory is given in [Bensoussan et al., 1993, Chapter 1, Section 4].

0.2.2 Cases Where $[\mathcal{D}(A), Y]_{\frac{1}{2}} = \mathcal{D}((-A)^{\frac{1}{2}})$

With reference to (0.10), we have that

$$[\mathcal{D}(A), Y]_{\frac{1}{2}} = \mathcal{D}\big((-A)^{\frac{1}{2}}\big); \text{ indeed,} \quad [\mathcal{D}(A), Y]_{1-\theta} = \mathcal{D}(A^{\theta}), \quad 0 < \theta < 1 \tag{0.15a}$$

(equivalent norms), in the following explicit and progressively more general cases:

(a) when A is a strictly negative, self-adjoint operator (in which case e^{At} is of negative type, and one may take $T = \infty$ in (0.9), (0.10), (0.14));

(b) when A is a normal generator, or it possesses a Riesz basis, on Y, and the fractional powers are well defined;

(c) when the closed operator A is maximal dissipative and $A^{-1} \in \mathcal{L}(Y)$ [Bensoussan et al., 1993, Proposition 6.1, p. 113]; thus identity (0.15) holds true for any generator of a s.c. contraction semigroup with $A^{-1} \in \mathcal{L}(Y)$. See Theorem A.2 in Appendix A of Chapter 3;

(d) when the closed operator $(-A)$ is positive, in the sense of [Triebel, 1978, Definition 1.14.1, p. 91], (in particular, A is the generator of a s.c. semigroup of negative type, by the Hille–Yosida theorem) and $(-A)$ has, locally, bounded imaginary powers: There exist two positive numbers ϵ and C such that $\|(-A)^{it}\|_{\mathcal{L}(Y)} \leq C$, for $-\epsilon \leq t \leq \epsilon$. For a proof that (0.15a) then holds true in this case, we refer to [Triebel, 1978, Theorem 1.15.3, p. 103]. One may also give a proof of (0.15a) by following the arguments of [Lions, Magenes, 1972, Theorem 14.1, p. 92], where one replaces the positive self-adjoint operator Λ in that proof with the present positive operator $(-A)$. Once (0.15a) is established, one obtains as a consequence, via the reinterpolation theorem, still for $0 < \theta < 1$:

$$[\mathcal{D}(A^{\alpha}), \mathcal{D}(A^{\beta})]_{\theta} = \mathcal{D}(A^{\gamma}), \quad \gamma = \alpha(1-\theta) + \beta\theta, \quad 0 \leq \alpha < \beta. \tag{0.15b}$$

The technical condition of local boundedness of imaginary powers is actually satisfied, in a general L_p context, in many cases of relevant classes of operators, including differential operators. The classes include:

(i) realizations of an elliptic differential operator, whose domain is defined by well-posed boundary conditions, with smooth coefficients [Seeley, 1971], [Fujiwara, 1969];

(ii) second-order elliptic operators with Holder-continuous coefficients [Prüss, Sohr, 1990];

(iii) negative generators of special semigroups [Clement, Pruss, 1990];

(iv) vector-valued ordinary differential operators of general order [Fuhrman, 1992];

(v) Stokes operator [Prüss, Sohr, 1991, Section 2];

(vi) m-dissipative operators A in Hilbert space with null space $\mathcal{N}(A) = 0$ [Prüss, Sohr, 1991, Section 2], using Nagy–Foias functional calculus [Sz-Nagy, Foias, 1970]; etc.

Cases of unbounded domains are also included. See also [Triebel, 1978].

0.2.3 Comments on the Proof of Proposition 0.1

Properties (0.9), (0.14)

Property (0.9) is most readily proved (in fact, with $T = \infty$), in the case where e^{At} is a s.c. analytic semigroup of negative type, by using: the Laplace (or Fourier) transform, the well-known λ-characterization $\|\lambda R(\lambda, A)\| \leq$ const of analyticity of the semigroup, and the Plancherel theorem. See [Lasiecka, 1980, Appendix A] for details, which are also reproduced in the proof of [Bensoussan et al., 1993, Lemma 3.3, p. 78]. This way, (0.9), and hence (0.14), are proved.

Property (0.10)

To prove property (0.10), one simply uses the "intermediate derivative theorem" [Lions, Magenes, 1972, Vol. 1, Theorem 2.3, p. 15] between $y \in L_2(0, T; \mathcal{D}(A))$ and $\dot{y} \in L_2(0, T; Y)$, previously established, to obtain (0.10).

Properties (0.11), (0.12)

These are easy and can be proved directly by making use of the analyticity property $\|(-A)^\theta e^{At}\| \leq c_T t^{-\theta}$, $0 < \theta < 1$, in the $\mathcal{L}(Y)$-norm, specialized to $\theta = 1/2 - \epsilon$ (L_2-kernel), and to $\theta = 1 - \epsilon$ (L_1-kernel), respectively.

Properties (0.13)

This is a highly nontrivial generalization of the case $p = 2$ in (0.9) and can be proved by using singular integrals [de Simon, 1964].

Remark 0.1 An additional case where *property* (0.11) *holds true with* $\epsilon = 0$ is when

$$\mathcal{D}\big((-A)^{\frac{1}{2}}\big) = \mathcal{D}\big((-A^*)^{\frac{1}{2}}\big).$$

Equivalently, one proves that $f \in L_2(0, T; [\mathcal{D}((-A)^{\frac{1}{2}})]')$ yields $y \in C([0, T]; Y)$ for (0.7). This is done by taking the Y-inner product of (0.7) with y, integrating over $[0, t]$, and using property (0.9), thus obtaining $y \in L_\infty(0, T; Y)$. By approximation, this is then boosted to $y \in C([0, T]; Y)$.

0.3 The Extrapolation Space $[\mathcal{D}(A^*)]'$

Let Y be a Hilbert space, as in this treatise, and let $A : Y \supset \mathcal{D}(A) \to Y$ be a closed operator, which is boundedly invertible: $A^{-1} \in \mathcal{L}(Y)$. Then $Y_1 \equiv \mathcal{D}(A)$ is a Hilbert space under the norm $\|x\|_{\mathcal{D}(A)} = \|Ax\|_Y$, which is equivalent to the graph norm. Set

$$Y_{-1} \equiv \text{completion of } Y \text{ under the (weaker) norm } \|A^{-1}x\|_Y. \tag{0.16}$$

Then, $Y_1 \subset Y \subset Y_{-1}$, with dense and continuous embeddings. We note that A is an isomorphism of $\mathcal{D}(A)$ onto Y, and A extends to an isomorphism of Y onto Y_{-1}. Moreover, the Y-adjoint A^* is an isomorphism from $\mathcal{D}(A^*)$ onto Y. By transposition, $(A^*)^*$ is an isomorphism of Y onto $[\mathcal{D}(A^*)]'$ where duality is with respect to the pivot space Y. Then, thanks to $(Ax, y)_Y = (x, A^*y)_Y$, we have $(A^*)^*x = Ax$, for $x \in \mathcal{D}(A)$, and $(A^*)^*$ is an *extension* of A, which we shall continue to denote by A: $(A^*)^* = A$: isomorphism from Y onto $[\mathcal{D}(A^*)]'$. Thus, we have $Y_{-1} = [\mathcal{D}(A^*)]'$. Throughout this treatise, we shall use the notation $\mathcal{D}(A)$ and $[\mathcal{D}(A^*)]'$ as in [Lions, Magenes, 1972], instead of Y_1 and Y_{-1}. The space $Y_{-1} = [\mathcal{D}(A^*)]'$ is an instance of an *extrapolation space* generated by A. A general theory for extrapolation spaces in a Banach space setting (replacing the Hilbert space Y) was introduced by Da Prato and Grisvard [1982; 1984] in 1982. See also Da Prato [1983].

0.4 Abstract Setting for Volume I. The Operator L_T in (1.1.9), or L_{sT} in (1.4.1.6), of Chapter 1

With reference to the abstract analytic (parabolic) class of Volume I (Chapters 1 through 6), our setting will essentially be as follows (modulo a translation of the generator):

(a) Let $(-A) : Y \supset \mathcal{D}(-A) \to Y$ (note the deliberate change of sign with respect to (0.7), and the remainder of this treatise, as well) be the generator of a s.c. analytic semigroup e^{-At} on Y, whose fractional powers A^θ, $0 < \theta < 1$, are well defined;
(b) If [ ]′ denotes duality with respect to the pivot space Y, let, for some $0 < \gamma < 1$:

$$B \in \mathcal{L}(U; [\mathcal{D}(A^{*\gamma})]'), \quad \text{or} \quad \text{equivalently, } B_1 \equiv A^{-\gamma} B \in \mathcal{L}(U; Y), \tag{0.17}$$

where $[\mathcal{D}(A^{*\gamma})]'$ is an extrapolation space (see point 0.3 above). We consider

$$y(t) = e^{-At} y_0 + (Lu)(t); \tag{0.18}$$

$$(Lu)(t) = \int_0^t e^{-A(t-\tau)} Bu(\tau)\, d\tau = \int_0^t A^\gamma e^{-A(t-\tau)} B_1 u(\tau)\, d\tau \tag{0.19a}$$

$$: \text{continuous } L_2(0, T; U) \to L_2(0, T; \mathcal{D}(A^{1-\gamma})), \tag{0.19b}$$

by (0.9), recalling (0.17). The $L_2(0, T; \cdot)$-adjoint L^* of L in (0.19) is

$$(L^* f)(t) = \int_t^T B^* e^{-A^*(\tau - t)} f(\tau)\, d\tau = \int_t^T B_1^* A^{*\gamma} e^{-A^*(\tau-t)} f(\tau)\, d\tau \tag{0.20a}$$

$$: \text{continuous } L_2(0, T; [\mathcal{D}(A^{1-\gamma})]') \to L_2(0, T; U). \tag{0.20b}$$

It readily follows from (0.19) (as in (0.12)) that, a fortiori,

$$u \in C([0, T]; U) \to Lu \in C([0, T]; Y). \tag{0.21}$$

However, if $1/2 \leq \gamma < 1$ and $u \in L_2(0, T; U)$, then $(Lu)(t)$ in (0.19) need *not* be in $C([0, T]; Y)$ in general; see an explicit, and readily constructed, counterexample (e.g., in [Li, Yong, 1995, p. 367]). Thus, in particular, $(Lu)(T)$ is generally nonmeaningful as an element of Y. There are two approaches that may be pursued.

(a) We may require that the time $t = T$ be a Lebesgue point for $(Lu)(t)$ and accordingly define

$$\begin{cases} \tilde{L}_T u \equiv \lim_{r \downarrow 0} \frac{1}{r} \int_{T-r}^{T} (Lu)(t)\, dt, \quad u \in \mathcal{D}(\tilde{L}_T); & (0.22) \\ \mathcal{D}(\tilde{L}_T) = \left\{ u \in L_2(0,T;U) : \lim_{r \downarrow 0} \frac{1}{r} \int_{T-r}^{T} (Lu)(t)\, dt \text{ exists in } Y \right\}. & (0.23) \end{cases}$$

Recalling (0.21), and recalling that if T is a point of continuity then a fortiori T is a Lebesgue point, we see that $C([0,T];U) \subset \mathcal{D}(\tilde{L}_T)$, and so $\mathcal{D}(\tilde{L}_T)$ is dense in $L_2(0,T;U)$. The adjoint operator $\tilde{L}_T^*$ of $\tilde{L}_T$ is given by (recalling $B_1 = A^{-\gamma}B \in \mathcal{L}(U;Y)$ from (0.17):

$$\begin{cases} (\tilde{L}_T^* y)(t) = B^* e^{-A^*(T-t)} y = B_1^* A^{*\gamma} e^{-A^*(T-t)} y; & (0.24) \\ \mathcal{D}(\tilde{L}^*) = \{ y \in Y : \tilde{L}_T^* y \in L_2(0,T;U) \} \begin{cases} = Y & \text{if } \gamma < \frac{1}{2}, \\ \supset \mathcal{D}(A^{*\gamma}) & \text{if } \frac{1}{2} \le \gamma < 1, \end{cases} & (0.25) \end{cases}$$

Since $\tilde{L}_T^*$ is densely defined, then $\tilde{L}_T$ is closable [Kato, 1996, p. 168]. The closure $\bar{\tilde{L}}_T$ of $\tilde{L}_T$ is the operator

$$\begin{cases} \bar{\tilde{L}}_T u = A^{\gamma - (\frac{1}{2} - \epsilon)} \int_0^T A^{\frac{1}{2} - \epsilon} e^{-A(T-t)} A^{-\gamma} B u(\tau)\, d\tau, & (0.26) \\ \mathcal{D}(\bar{\tilde{L}}_T) = \left\{ u \in L_2(0,T;U) : \int_0^T A^{\frac{1}{2} - \epsilon} e^{-A(T-t)} A^{-\gamma} B u(\tau)\, d\tau \right. \\ \qquad \left. \in \mathcal{D}\left(A^{\gamma - (\frac{1}{2} - \epsilon)}\right) \right\}, & (0.27) \end{cases}$$

where $\epsilon > 0$ and the integral in (0.26), or (0.27), is well defined by (0.11).

(b) Recalling (0.21), one may require more and set

$$\begin{cases} L_T u = \lim_{t \uparrow T} (Lu)(t) = \lim_{t \uparrow T} \int_0^t e^{-A(t-\tau)} B u(\tau)\, d\tau, & (0.28) \\ \mathcal{D}(L_T) = \left\{ u \in L_2(0,T;U) : \lim_{t \uparrow T} (Lu)(t) \text{ exists in } Y \right\}. & (0.29) \end{cases}$$

By (0.21), $C([0,T];U) \subset \mathcal{D}(L_T)$ and thus L_T is *densely defined* in $L_2(0,T;U)$. The adjoint L_T^* of L_T is the same as $\tilde{L}_T^*$ in (0.24), (0.25): $L_T^* \equiv \tilde{L}_T^*$. Consider

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the following operator

$$\begin{cases} A^\gamma K_T u, & (0.30) \\ \mathcal{D}(A^\gamma K_T) = \Big\{ u \in L_2(0,T;U) : K_T u \\ \quad \equiv \int_0^T e^{-A(T-t)} A^{-\gamma} B u(\tau)\, d\tau \in \mathcal{D}(A^\gamma) \Big\}. & (0.31) \end{cases}$$

Notice that $A^\gamma K_T$ is closed, being the composition of the bounded operator K_T in (0.31), followed by the closed operator A^γ with bounded inverse $A^{-\gamma} \in \mathcal{L}(Y)$ [Kato, 1996, p. 164]. Moreover, we have

$$L_T u = A^\gamma K_T u, \quad u \in \mathcal{D}(L_T), \tag{0.32}$$

so that L_T is closed and $A^\gamma K_T$ is an extension of L_T. To establish (0.32), one way is to apply $A^{-\gamma} \in \mathcal{L}(Y)$ to both sides of definition (0.28) with $u \in \mathcal{D}(L_T)$, move $A^{-\gamma}$ inside the limit and the integral, and obtain

$$\begin{aligned} A^{-\gamma} L_T u &= \lim_{t \uparrow T} \int_0^t e^{-A(t-\tau)} A^{-\gamma} B u(\tau)\, d\tau \\ &= \lim_{t \uparrow T} K_t u = K_T u \in Y, \quad u \in \mathcal{D}(L_T), \end{aligned} \tag{0.33}$$

where K_T is defined in (0.32), and where

$$K_t u \equiv \int_0^t e^{-A(t-\tau)} A^{-\gamma} B u(\tau)\, d\tau \in C([0,T];Y). \tag{0.34}$$

Then (0.33) leads to (0.32).

In the treatment of the optimal control problem in Chapter 1, one may pursue either one of the above approaches (a) or (b). We shall explicitly follow (b) and use definition (0.28), (0.29) for L_T.

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