SPECIALITY ONE RATIONAL SURFACES IN $\mathbb{P}^4$

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Introduction: We work over an algebraically closed field **k** of characteristic zero, except in section (3) where the characteristic is arbitry. By a surface we will mean a smooth projective surface and a curve will be any effective divisor on a surface. We recall that in [A], the speciality of a rational surface X in $\mathbb{P}^n$ is defined to be the number $q(1)=h^1(\mathcal{O}_X(H))$, where H is a hyperplane section of X. We say that X is special or non-special in accordance with $q(1)>0$ or $q(1)=0$.

In [A], a complete classification of non-special rational surfaces in $\mathbb{P}^4$ was given, showing that the linearly normal ones form, for each degree $3\leq d\leq 9$, a single irreducible family. Recently in [E-P] it was shown that there are only a finite number of irreducible components of the Hilbert scheme of $\mathbb{P}^4$ containing rational surfaces; in particular the degrees of such surfaces is bounded. The results which we present here are a contribution to the eventual determination of all such components and contributes to the classification of surfaces in $\mathbb{P}^4$ of small degree [A], [A-R], [R], [Ro],and varieties with small invariants $[I_1,I_2,I_3]$.

We will be concerned with rational surfaces of speciality one in $\mathbb{P}^4$. By $[O_1,O_2,O_3]$ these have degree eight or more and a simple argument shows that their degree is at most eleven (prop.(1.1)). By $[O_3]$, those in degree eight form a single irreducible family and in degree ten we have the following theorem from [R]:

Theorem:(Ranestad) *Let X be a speciality one rational surface of degree ten in $\mathbb{P}^4$, then X is a blowing up of $\mathbb{P}^2$ in thirteen points $x_1,\ldots,x_{13}$, embedded in $\mathbb{P}^4$ by the linear system*

$$|D|=|\pi^*14L-6E_1-4E_2-\ldots-4E_{10}-2E_{11}-E_{12}-E_{13}|$$

where $\pi:X\to\mathbb{P}^2$ is the blowing up in the x_i's, L is a line in $\mathbb{P}^2$ and E_i is the fiber of π over x_i. /

In §(5) we will prove the following theorem:

Theorem(1): *The Hilbert scheme of speciality one, degree ten, rational surfaces in $\mathbb{P}^4$ is irreducible of dimension 38. /*

In degree nine we have the following:

Theorem(2): *(a) Let X be a rational surface of degree nine in $\mathbb{P}^4$ with speciality $q(1)>0$, then $q(1)=1$ and the first and second adjunctions of X give a canonical sequence of birational morphisms of rational surfaces*

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2$$

where X_2 is canonically a cubic surface in $\mathbb{P}^3$. The morphism f_2 blows up in three distinct closed points x_7, x_8, x_9, while f_1 blows up in six distinct closed points $x_{10},\ldots,x_{15}$. Letting K_1 and K_2 be the inverse images on X of the canonical divisors on X_1 and X_2 respectively, the linear system of hyperplane sections of X is given by

$$|H| = |-K-K_1-K_2|$$

where K is the canonical divisor on X.

(b) The Hilbert scheme of special rational surfaces of degree nine in $\mathbb{P}^4$ is irreducible of dimension 42. /

Remark: If we let $f_3: X_2 \to \mathbb{P}^2$ be one of the finitely many expressions for X_2 as $\mathbb{P}^2$ blown up in six points $x_1,\ldots,x_6$, we obtain an expression for X as $\mathbb{P}^2$ blown up in fifteen points $x_1,\ldots,x_{15}$, with

$$|H| = |\pi^*9L-3E_1-\ldots-3E_6-2E_7-2E_8-2E_9-E_{10}-\ldots-E_{15}|. /$$

The difficulty which one confronts in trying to show that the Hilbert scheme is irreducible, arises from the speciality of the surface. As was explained in [A] the speciality of the surface X is reflected in the special position of the blown up points of $\mathbb{P}^2$. As in the non-special case we construct a parameter family for special rational surfaces, as a universal family of blowings up of $\mathbb{P}^2$. In the non-special case the irreducibility is an automatic consequence of this construction,since the universal variety of ordered blowings up of $\mathbb{P}^2$ is itself irreducible. However in the special case, we are working over a closed subset of this variety which is not apriori irreducible. To treat this phenomena, we are obliged to establish certain properties of the linear systems determining the special position of the points, for all the surfaces in the family. In the case of degree nine, our objective is to show that every configuration of fifteen points, giving rise to a surface in $\mathbb{P}^4$ via the linear system $|H|$ of theorem(2), is a specialisation of the following generic configuration:

Proposition(3): *Let $x_1,\ldots,x_9$, be nine generic closed points of $\mathbb{P}^2$ and let $\mathcal{P}$ be the generic pencil of plane sextics which are singular at $x_1,\ldots,x_6$ and tangent at x_7,x_8,x_9. Then the base of $\mathcal{P}$ contains six further distinct closed points $x_{10},\ldots,x_{15}$. The generic special rational surface of degree nine in $\mathbb{P}^4$ is then obtained by blowing up the x_i's and embedding by the linear system $|H|$ given in theorem(2).* /

This leaves open the question of speciality one, rational surfaces of degree eleven in $\mathbb{P}^4$ as much for the existence as for the irreducibility of the family.

SECTION 1: Uniqueness.

In this section we use freely the general theory on the adjunction mapping (see [S],[So],[V]).

Proof : (of theorem (2)(a)).

Firstly we will show that if q(1)=1, then X is of the form indicated in the theorem. Let H be the general hyperplane section of X in $\mathbb{P}^4$ and let K be the canonical divisor on X. Then by [A] section (5), we have

$$H^2 = 9 \ , \ H.K = 3 \ , \ K^2 = -6 \ , g = 7$$

where g is the genus of H. By [So] (1.5) and (3.1), the linear system $|H+K|$ is without base points and the resulting morphism $\phi : X \to \mathbb{P}^6$ induces a birational morphism $f_1 : X \to X_1$ of X to its smooth image X_1 in $\mathbb{P}^6$, so that f_1 blows up X_1 in a finite number s_1 of distinct closed points.

Let H_1 be the general hyperplane section of X_1 in $\mathbb{P}^6$. Then we have

$$H_1{}^2 = 9 \ ; \ H_1 . K_1 = -3 \ ; \ K_1{}^2 = -6 + s_1 \ ; \ g_1 = 4 \ ,$$

where g_1 is the genus of H_1. Once again by [So] (1.5), $|H_1+K_1|$ is without base points and induces a morphism $\phi_1 : X_1 \to \mathbb{P}^3$.

By the general theory of the adjunction mapping, we have several cases to consider. Firstly, since $|H_1+K_1|$ has no base points we have

$$0 \leq |H_1+K_1|^2 = -3+s_1$$

Elimination of the case $s_1 = 3$:

If $s_1 = 3$, then by [So] (2.1), the image of X_1 by ϕ_1 is a smooth twisted cubic in $\mathbb{P}^3$. Letting $g : X_1 \rightarrow \mathbb{P}^1$ be the induced morphism of X_1 to its image, the singular fibers of g are of the form $P_1 + P_2$ with $P_i \cong \mathbb{P}^1$ (i = 1, 2) and $P_1^2 = -1 = P_2^2 = -P_1 . P_2$.

Blowing down P_1 in each singular fiber we obtain a factorisation

$$X_1 \xrightarrow{g_1} F_e \xrightarrow{g_2} \mathbb{P}^1 \qquad (e \geq 0)$$

of g, where g_1 expresses X_1 as the blowing up of a Hirzebruch surface F_e ([H],(v)) in eleven points and g_2 is the canonical map expressing F_e as a fibered projective line over $\mathbb{P}^1$.

Now note that we can suppose e = 1. This is clear if e = 0 or 1. If e≥2, let C_0 be the unique section of g_2 with negative self intersection ($C_0^2 = -e$) and let λ be the number of blown up points which lie on C_0. Let C_0' be the strict transform of C_0 on X_1, so that $C_0^2 = - e - \lambda$.Then using the facts

$$H_1+K_1 = g_1^*(3f) \qquad \text{(f one fiber of } g_2)$$

$$K_1 = g_1^*(-2C_0 -(e + 2)f)-E_1- \dots - E_{11}$$

where E_i (i = 1, ..., 11) are the blown down curves of the singular fibers, we conclude that $0 < H_1 . C_0 = 5 - e - \lambda$, since H_1 is very ample and C_0 is effective.

This shows that $\lambda \leq 4 - e$ and we can suppose that $E_1, \dots, E_{e-1}$ don't meet C_0. Finally by exchanging E_i for its complementary component in the singular fiber of g, for i = 1, ..., e-1, we obtain the desired factorisation through F_1. In fact the complementary components are the strict transforms of fibers of g_2, so they meet C_0 in just one point. After blowing down we obtain a section over $\mathbb{P}^1$ with self intersection -1.

Now we can project F_1 to $\mathbb{P}^2$ to obtain a birational morphism $p : X_1 \rightarrow \mathbb{P}^2$ which blows up $\mathbb{P}^2$ in twelve points $x_1, \dots, x_{12}$ so ordered that H_1 takes the form $H_1 = p^*6L-4E_1-E_2-\dots-E_{12}$ .Letting $\pi = p \circ f_1$, we have

$$H = \pi^*9L-5E_1-2E_2-\dots-2E_{12}-E_{13}-E_{14}-E_{15}.$$

Now we will show that H is not very ample. There is at least one curve C in the linear system $|\pi^*4L-2E_1-E_2-\dots-E_{12}|$ and for such C we have $H . C = 4$, $p_a(C) = 2$. However there are no projective curves, reducible or not of arithmetic genus two and degree four.

We have now shown that $s_1 > 3$. In this case $(H_1 + K_1)^2 > 0$ so that by [So] (2.3), the image X_2 of X_1 by ϕ_1 is a hypersurface of $\mathbb{P}^3$ of degree two or more. By [So] (2.3) and (3.1), except for the case $K_1^2 = 1$ ($s_1 = 7$) corresponding to the Bertini Involution [So] (2.5.2), the image X_2 of X_1 is a linearly normal rational surface in $\mathbb{P}^3$. In the latter case there are only two possibilities ; X_2 is a smooth quadric ($s_1 = 5$) or a smooth cubic surface ($s_1 = 6$).

Elimination of the case $s_1 = 7$.

In this case, the induced morphism $f_2 : X_1 \to X_2$ of X_1 to its image in $\mathbb{P}^3$, is the Bertini involution ; so we have a birational morphism [D] (p.66) $p : X_1 \to \mathbb{P}^2$ expressing X_1 as $\mathbb{P}^2$ blown up in eight points $x_1, ..., x_8$, H_1 takes the form $H_1 = p^*6L-2E_1-...-2E_8$. Letting $\pi = p \circ f_1$ we find

$$H = \pi^*9L-3E_1- ... -3E_8-E_9- ... -E_{15} .$$

Now in the linear system $|\pi^*3L-E_1-...-E_9|$ there is a curve C with H.C =2 and p_a (C) = 1, showing that H is not very ample.

Elimination of the case where $s_1 = 5$

In this case, the induced morphism $f_2 : X_1 \to X_2$ of X_1 to its image in $\mathbb{P}^3$ is the blowing up of the smooth quadric X_2, in nine points. Blowing up a point on X_2 and blowing down to $\mathbb{P}^2$, we obtain a birational morphism $p : X_1 \to \mathbb{P}^2$ expressing X_1 as $\mathbb{P}^2$ blown up in ten points $x_1, ..., x_{10}$ and H_1 takes the form $H_1 = p^*5L-2E_1-2E_2-E_3-...- E_{10}$. Letting $\pi = p \circ f_1$, we have $H = \pi^*8L-3E_1-3E_2-2E_3-...- 2E_{10}-E_{11}-...-E_{15}$. Now let C be a curve in the linear system $|\pi^*4L-2E_1-2 E_2-E_3- ... -E_{10}|$ .Then we have H . C = 4 and $p_a(C) = 1$, so that C is contained in a hyperplane of $\mathbb{P}^4$. This shows that the linear system $|H - C| = |\pi^*4L - E_1 - ... - E_{15}|$contains a curve D with H . D = 5 and p_a (D) = 3. Such a curve is necessarily the union of a plane quartic D_1 and a line Δ, meeting D_1 in a point. We then have D_1 in a linear system $|\pi^*dL - r_1E_1 - ... - r_{15} E_{15}|$ with $d \leq 4$. Since p_a $(D_1) = 3$, we have $d = 4$ and $r_i \leq 1$, but this implies that Δ is not effective.

The case $s_1 = 6$

In this case, the induced morphism $f_2 : X_1 \to X_2$ of X_1 to its image in $\mathbb{P}^3$, is the blowing up of the smooth cubic surface $X_2 \subset \mathbb{P}^3$. The hyperplane section of X_2 is the anticanonical system giving $|H| = |-K - K_1 - K_2|$ in the notation of the theorem. Now it is well known that every smooth cubic surface in $\mathbb{P}^3$ is a blowing of $\mathbb{P}^2$ in six distinct closed points in one of finitely many ways [H] (v).

The fact that there are no rational surfaces of degree nine in $\mathbb{P}^4$ with q(1)>1, follows from ([A-R] §1). /

Proposition(1.1): *There are no speciality one rational surfaces in $\mathbb{P}^4$ of degree twelve or more.*

preuve: In fact in this case we would have |-K| effective with H.(-K)<0 which is impossible.

SECTION 2: RESULTS ON LINEAR SYSTEMS.

In this section we will establish numerous results about certain linear systems on a special rational surface of degree nine in $\mathbb{P}^4$, and a number of results on very ampleness. The reader is advised to consult this section as a reference for the proofs of theorems in later sections.

We let X be a special rational surface of degree nine in $\mathbb{P}^4$, then, as indicated in theorem (2)(a) and the remark which follows it, we have a birational morphism $\pi : X \to \mathbb{P}^2$ expressing X as $\mathbb{P}^2$ blown up in fifteen points $x_1, ..., x_{15}$ by three successive blowings up. We can thus write

$$K = \pi^*(-3L)+E_1+ ... +E_{15}$$
$$K_1 = \pi^*(-3L)+ E_1+ ... +E_9$$
$$K_2 = \pi^*(-3L)+E_1+ ... + E_6$$

As before, H is a general hyperplane section of X.

The important lemmas (2.4), (2.7), (2.8) allow us to stratify the parameter scheme $\mathcal{H}_1$ which will be constructed in §4. The first strate $\mathcal{H}_2 \subset \mathcal{H}_1$ is given by the equivalent conditions of lemma (2.7) and the second $\mathcal{H}_3 \subset \mathcal{H}_2$ is given by the conditions of lemma (2.8) . We show that the open subset $\mathcal{H}_2 - \mathcal{H}_3$ of $\mathcal{H}_2$ is irreducible and that $\mathcal{H}_3$ is the union of three disjoint irreducible components W'(i) (i=7,8,9,) corresponding to the cases i=7,8,9 of lemma (2.8). We then show by an indirect argument from deformation theory , that the generic members of $\mathcal{H}_2$ et W(i) (i=7,8,9) are degenerations of the generic member of $\mathcal{H}_1$ described in proposition (3).

Lemma (2.1): *We have* $h^\circ(\mathcal{O}(-K_1)) = 1$ *and* $h^\circ(\mathcal{O}(-K-K_2)) = 2$.

Proof : It is clear that $h^\circ(\mathcal{O}(-K_1)) \geq 1$. Let C be an effective divisor in $|-K_1|$, then we have the canonical exact sequence :

$$o \to \mathcal{O}(-K-K_2) \to \mathcal{O}(H) \to \mathcal{O}_C(H) \to o$$

and, since H.C = 3 and $p_a(C)=1$, it follows that the image of the canonical map $H^\circ(\mathcal{O}(H)) \to H^\circ(\mathcal{O}_C(H))$ is three dimensional. Since $h^\circ(\mathcal{O}(H)) = 5$, this gives $h^\circ(\mathcal{O}(-K-K_2)) =2$.

Now let D be an effective divisor in $|-K-K_2|$, then, since H.D =6 and p_a (D) = 4, it follows that the image of the canonical map $H^\circ(\mathcal{O}(H)) \to H^\circ(\mathcal{O}_D(H))$ is at least four dimensional. This gives $h^\circ(\mathcal{O}(-K_1)) = 1$.

Definition (2.2) : From now on, we let C be the unique curve in $|-K_1|$ and we let D be the generic curve in $|-K-K_2|$. The following lemma shows that C has no multiple components.

Lemma (2.3) : *Let* Δ *be an effective divisor on X such that* $H.\Delta = 3$ *and* p_a (Δ)

= 1, then Δ is a plane cubic in $\mathbb{P}^4$ without multiple components.

Proof : It is immediate that Δ is a plane cubic and since no plane curve on a smooth projective surface can have a multiple component, this proves the lemma. /

We now investigate what happens if C is reducible.

Lemma (2.4) : *Suppose C is a reducible curve, C = A + B, where A is a line and B is a conic. The following two cases are the only ones possible :*

(i) $A^2 = -2$, $B^2 = -2$

(ii) $A^2 = -3$, $B^2 = -1$

In case (ii), B is smooth and B is one of the three curves E_7, E_8, E_9.

Proof : (The first part is trivial). By the two connexity of |H| we conclude that $A^2 \le -1$.If $A^2 = -1$,then A is one of the curves $E_{10}, \ldots, E_{15}$. The latter would imply that B = C - A had arithmetic genus one contradicting the fact that B is a conic. So we have $A^2 \le -2$. If $B = B_1 + B_2$ is reducible, then the same argument shows that $B_1{}^2 \le -2$ and $B_2{}^2 \le -2$ implying that $B^2 = B_1{}^2 + B_2{}^2 + 2 \le -2$. If B is smooth, then B is birational to its image by the first adjunction on X so 0<(H+K).B = 2+B.K and by the adjunction formula we conclude that $B^2 \le -1$. Now, noting that $C^2 = 0$ and A.B = 2, we have $A^2 + B^2 = -4$ which gives the two cases and shows that B is smooth in case (ii).

Now suppose that B is smooth and that $B^2 = -1$. Then (H + K).B = 1 and since, as seen above, B does not meet $E_{10}, \ldots, E_{15}$ we conclude that by the first adjunction on X, the image of B is a line on X_1 with self intersection -1. This shows that B is one of the curves E_7, E_8, E_9.

Lemma (2.5) : *The linear system |D| is a complete pencil and the generic curve D in |D| is smooth of genus four. The base of |D| is smooth of degree $D^2 = 3$. Equally, we have $\mathcal{O}_D(H) = \omega_D$ the canonical sheaf on D and*

$$\mathcal{O}_D(D) = \mathcal{O}_D(E_7 + E_8 + E_9)$$

so that the base of the pencil |D| is formed by the three closed points $y_i = D \cap E_i$; $i = 7, 8, 9$.

Proof : We note firstly that if |D| has no fixed components, then since $D^2 = 3$, no two curves of |D| have a singularity at a point in the base of |D| and using Bertini's theorem in characteristic zero we can conclude that the generic curve in |D| is smooth and this will prove the first part of the lemma. We note firstly that any fixed component of |D| is contained in the plane of C.

Step (i) : No fixed component of |D| is a component of C.

If C was a fixed component of |D| , then |H - 2C| = |D-C| would be a complete pencil and 2C would be a plane curve of degree H.(2C) = 6 and arithmetic genus $p_a(2C) = 1$, but this is impossible. If C = A + B and A is a fixed component of $|D|$, then, as above, $C + A = 2A + B$ is a plane curve, but as in the proof of lemma (2.3), $2A$ is not a plane curve.

Step (ii) : $|D|$ has no fixed components that are not components of C.

Let Δ be the fixed curve of $|D|$. Since Δ and C are in a common plane and have no components in common we have $\Delta.C \geq 3$. Since $|D - \Delta|$ has no fixed components we have $3-\Delta.C = (D-\Delta).C \geq 0$. This gives $\Delta . C = 3$ and Δ is a line in $\mathbb{P}^4$.

Now since the complete pencil $|D - \Delta|$ has no fixed components, it follows that the generic curve D_0 in $|D - \Delta|$ is integral and the fact $D_0 . C = 0$ implies that there are disconnected curves in $|D_0 + C| = |H - \Delta|$. By [So](0.10.1), $|H - \Delta|$ is without base points and we conclude that $(H - \Delta)^2 = 0$. A simple calculation then gives $\Delta^2 = -7$, $\Delta . D = -2$, $p_a(D_0) = 0$.

Finally, as in the proof of [So] (1.1.1) , the morphism $X \to \mathbb{P}^2$ induced by the linear system $|H - \Delta|$ factorises as

$$X \xrightarrow{r} \mathbb{P}^1 \xrightarrow{s} \mathbb{P}^2$$

where s embeds $\mathbb{P}^1$ as a conic and r has connected fibers. Since the connected curves C and D_0 lie in separate fibers of the flat morphism r we should have $p_a(D_0) = p_a(C) = 1$, which is a contradiction.

We have now shown that the generic curve D in $|D|$ is smooth and since $H . D = 6$ and $p_a(D) = 4$, it follows that D is a canonical curve of degree six and genus 4 in a hyperplane of $\mathbb{P}^4$. This shows that $\mathcal{O}_D(H) = \omega_D$ on D. Noting that $\omega_D = \mathcal{O}_D(D + K)$, a simple manipulation gives $\mathcal{O}_D(D) = \mathcal{O}_D(E_7 + E_8 + E_9)$ and, since x_7, x_8, x_9 are distinct closed points of X_1 (notation of theorem (2)), the curves E_i ($i = 7, 8, 9$) have no points in common. Noting that $D . E_i = 1$ ($i = 7, 8, 9$) this completes the proof. /

Lemma (2.6) : *For $i = 7, 8, 9$, we have $h^\circ(O(D-E_i)) = 1$, so that the linear system $|D-E_i|$ contains a unique curve D_i.*

Proof : In fact $y_i = D \cap E_i$ is, by lemma (2.5), a fixed point of the complete pencil $|D|$, so considering the cohomology of the followingcanonical exact sequence

$$o \to \mathcal{O}(D-E_i) \to \mathcal{O}(D) \to \mathcal{O}_{E_i}(D) \to o$$

we see that the canonical map, $H^\circ(\mathcal{O}(D)) \to H^\circ(\mathcal{O}_{E_i}(D))$ has a one dimensional image. This gives $h^\circ(\mathcal{O}(D-E_i)) = 1$. /

Lemma (2.7) : *Let Δ be the base of the pencil $|D|$ and let $G = H - 2C$, then the following conditions are equivalent :*

(i) the canonical map , $H^\circ(\mathcal{O}(H + K)) \to H^\circ(\mathcal{O}_\Delta(H + K))$ is not

surjective.

(ii) the canonical map , $H^\circ(\mathcal{O}_D(H+K)) \to H^\circ(\mathcal{O}_\Delta(H+K))$ is not surjective.

(iii) $\mathcal{O}_D(E_{10}+\ldots+E_{15}) = \omega_D$

(iv) $\mathcal{O}_D(D) = \mathcal{O}_D(C)$

(v) $\mathcal{O}_D(G) = \mathcal{O}_D$

(vi) The divisor G on X is effective.

what is more, if the above conditions are verified then $h^\circ(O(2C)) = 2$.

Proof :(i)⇔(ii). This results from a consideration of the cohomology of the canonical exact sequence

$$0 \to \mathcal{O}(E_{10}+\ldots+E_{15}) \to \mathcal{O}(H+K) \to \mathcal{O}_D(H+K) \to 0$$

noting that $h^1(\mathcal{O}(E_{10}+\ldots+E_{15})) = 0$.

(ii)⇔(iii). The ideal sheaf of Δ, as a divisor on D, is $\mathcal{O}_D(-D)$, so we have the following canonical exact sequence

$$0 \to \mathcal{O}_D(E_{10}+\ldots+E_{15}) \to \mathcal{O}_D(H+K) \to \mathcal{O}_\Delta(H+K) \to 0$$

Since (H + K) . D = 9, we conclude that H + K is non-special on D. This shows that we have (ii) if and only if $h^1(\mathcal{O}_D(E_{10}+\ldots+E_{15})) > 0$ and by Clifford's theorem, this is equivalent to (iii), since in any case $h^\circ(\mathcal{O}_D(E_{10}+\ldots+E_{15})) \geq 3$.

(iii) ⇔ (iv) ⇔ (v) These equivalences result by simple manipulation noting

(v) ⇔ (vi) We have the exact sequence $0 \to \mathcal{O}(-C) \to \mathcal{O}(G) \to \mathcal{O}_D(G) \to 0$ with $h^1(\mathcal{O}(-C)) = h^1(\mathcal{O}(C+K)) = h^1(\mathcal{O}(E_{10}+\ldots+E_{15})) = 0$ and $h^\circ(\mathcal{O}(-C)) = 0$. This shows that $H^\circ(\mathcal{O}(G)) \approx H^\circ(\mathcal{O}_D(G))$.

Since G . D = 0, we have $h^\circ(\mathcal{O}_D(G)) > 0$ if and only if $\mathcal{O}_D(G) \approx \mathcal{O}_D$ and this gives the desired equivalence. Now suppose that G is effective, then from the exact sequence $0 \to \mathcal{O}(2C) \to \mathcal{O}(H) \to \mathcal{O}_G(H) \to 0$ and the facts H . G = 3, $p_a(G) = 1$, we conclude that the image of the canonical map $H^\circ(\mathcal{O}(H)) \to H^\circ(\mathcal{O}_G(H))$ is three dimensional. This gives $h^\circ(\mathcal{O}(2C)) = 2$. /

Lemma (2.8) : *If the conditions of lemma (2.7) are verified and the complete pencil |2C| is not base point free, then C = A + B where H . A = 1 , $A^2 = -3$ and B is the fixed curve of |2C| with H . B = 2 ; $B^2 = -1$. In particular B is one of the curves E_i (i = 7, 8, 9) and A is smooth and irreducible.*

Proof : Since $C^2 = 0$, if |2C| is not base point free, it has a fixed component whose integral components are components of C. Since $h^\circ(\mathcal{O}(C)) = 1$ and $h^\circ(\mathcal{O}(2C)) = 2$ it follows that C is not a fixed component of |2C|.

As such we can write C = A + B where A is a component of the fixed curve of |2C| and B has no components in common with the fixed curve of |2C|. As such,

$$|2C - A| = |A + 2B|$$

is a complete pencil and $0 \leq (A + 2B) . B = 2 + 2B^2$, so that $B^2 \geq -1$. Now by lemma (2.4) we conclude that $B^2 = -1$ and that B is a smooth conic in $\mathbb{P}^4$ with $B = E_i$ (i = 7, 8, 9). Finally | 2C | cannot have 2A as fixed component, otherwise |2B| would be a pencil without fixed components and $(2B)^2 \geq 0$, but this contradicts $B^2 = -1$. /

Lemma (2.9) : *Let Δ be the image on X_1 of the pencil | D | on X, let A be the base of Δ and let B be the base of | D |. Then*

(i) The induced morphism $B \to A$ is a closed immersion

(ii) A contains the closed subscheme $\{ x_{10} , \dots , x_{15} \}$

(iii) A is curvelinear.

Let $f_0 : X_0 \to X_1$ be the blowing up of X_1 in the image of B and let Δ_0 bethe strict transform on X_0 of the pencil Δ. Then the base of Δ_0 is canonically isomorphic to Δ.

Proof :This results from the fact that $x_{10} , \dots , x_{15}$ are distinct closed points of X_1 and any one of the points y_i (of lemma (2.5)) is infinitely closed to at most one of the points $x_{10} , \dots , x_{15}$. /

Proposition (2.10): *Let X be a surface and let |H| be a complete linear system on X. We suppose that there is a curve C and a pencil |D| on X, such that C+|D| $\subset$ |H|. If the following conditions are verified*

(1) The canonical map, $H^0(\mathcal{O}(H)) \to H^0(\mathcal{O}_C(H))$ is surjective

(2) The canonical map, $H^0(\mathcal{O}(H)) \to H^0(\mathcal{O}_D(H))$ is surjective for every curve D in |D|,

(3) $\mathcal{O}(H)$ is very ample on C and on every curve in |D|, then the linear system |H| is very ample on X.

Proof: Let Z be some finite degree two closed subscheme of X. If Z is contained in C or one of the curves in |D|, then |H| separates Z by (1) and (2). We can thus suppose that Z meets C, or a curve in |D|, in at most a closed point. In particular we can suppose that Z does not meet the base of the pencil |D|. If Z meets C, then C+D separates Z for a general choice of D in |D|. If Z does not meet C then some curve D' in |D| meets Z in a closed point and C+D' separates Z. /

Proposition (2.11) : *Let $C = C_1 + C_2$ be the sum of two effective divisors on a projective surface X and let $\mathcal{L}$ be an invertible sheaf on C. We suppose that the following three conditions are verified:*

(a) The canonical map, $H^\circ(\mathcal{L}) \to H^\circ(\mathcal{L} \otimes_C \mathcal{O}_{C_i})$ (i = 1,2) is