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137 Metric Diophantine Approximation on Manifolds

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Metric Diophantine Approximation on Manifolds

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To Haleh Afshar and Tatiana Bernik

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Preface

This book is about metric Diophantine approximation on smooth manifolds embedded in Euclidean space. The aim is to develop a coherent body of theory on the lines of that which already exists for the classical theory, corresponding to the manifold being Euclidean space. Although the functional dependence of the coordinates presents serious technical difficulties, there is a surprising degree of interplay between the very different areas of number theory, differential geometry and measure theory.

A systematic theory began to emerge in the mid-1960's when V. G. Sprindžuk and W. M. Schmidt established that certain types of curve were extremal (an extremal set enjoys the property that, in a sense that can be made precise, Dirichlet's theorem on simultaneous Diophantine approximation cannot be improved for almost all points in the set; thus the real line is extremal). Sprindžuk conjectured that analytic manifolds satisfying a necessary nondegeneracy condition are extremal. Over the last 30 years, there has been considerable progress in verifying this conjecture for manifolds satisfying various arithmetic and geometric constraints, culminating in its recent proof by D. Y. Kleinbock and G. A. Margulis using ideas of flows on homogeneous spaces of lattices [139]. The greater part of this book is concerned with establishing the counterparts of Khintchine's theorem for manifolds and with the Hausdorff dimension of the associated exceptional sets. It relies very much on Sprindžuk's important monographs *Mahler's problem in metric number theory* [208] and *Metric theory of Diophantine approximations* [210]; indeed to some extent it can be regarded as a sequel. Our approach, like Sprindžuk's, is largely analytic and geometric and flows on lattices are not used nor, apart from the last chapter, is ergodic theory. These approaches, however, hold great promise even for the more delicate questions of Khintchine type results and Hausdorff dimension on manifolds.

Chapter 1 sets out the background required for metric Diophantine approximation on manifolds. Khintchine's theorem on simultaneous Diophantine approximation and its dual form (Groshev's theorem) are considered for manifolds in Chapter 2, which is devoted mainly to the long and demanding proof of a closely related conjecture of A. Baker concerning the rational normal curve $\{(t, \dots, t^n) \colon t \in \mathbb{R}\}$. Chapter 3 begins with a relatively self-contained account of Hausdorff dimension and an introduction to its uses in Diophantine approximation. A fuller discussion is given in Chapters 4 and 5 which deal respectively with the technically different problems of obtaining upper and lower bounds. The range of techniques from the number theory arsenal which are called upon are an indication of the level of difficulty of some of the questions. The p-adic case is discussed fairly briefly in Chapter 6; the final chapter is devoted to various applications of metric Diophantine approximation.

Theorems, lemmas and so on are numbered consecutively in each chapter. Sections are denoted § $l.m$ and subsections by § $l.m.n$. The scope and complexity of the material has made notation something of a problem and to help the reader a list precedes Chapter 1. While not complete, the references are nevertheless intended to be reasonably comprehensive and include less well known papers from the former Soviet Union.

It is with sadness that we record that our friend and colleague Yuri Melnichuk would have been an author but for his tragic death during a visit to York in 1993. We are very grateful to many people and particularly to Haleh Afshar and Tatiana Bernik for their support and encouragement during this setback and throughout the book's lengthy gestation. Alan Baker has given us constant encouragement, Victor Beresnevich, Detta Dickinson, Sanju Velani, James Vickers and Chris Wood read parts of earlier drafts and made many suggestions and corrections; Peter Jackson read the proofs and removed numerous inconsistencies and typographical errors. They are not, however, responsible for any mistakes remaining.

The book was prepared on a Silicon Graphics Personal Iris workstation using LaTeX and GNU Emacs installed by Michael Beaty who with Simon Eveson sorted out our TeX problems with skill and good humour. Roger Astley of the Cambridge University Press has been patient and understanding beyond the call of duty. The collaboration essential to this book would not have been possible without the support that the Royal Society and the Soros Foundation provided for exchanges between the Belorussian Academy of Sciences at Minsk, the Lvov Polytechnic Institute and the University of York. The help which we have had has been invaluable and has ensured that this book will be published this side of the millennium.

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